

On the Independent Crossing Approximation in Three Flavor Neutrino Oscillations

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Abstract

In dense matter like the Supernovae and at high energy, neutrinos have two points where the nonadiabatic transitions could occur. With the present values of the oscillation parameters in the standard three flavor scenario, two nonadiabatic transitions can be regarded as independent, i.e., two can be treated by the two flavor formalism separately. However, in the presence of new physics, it is not clear in general whether such independence holds. We examine this question by assuming hypothetical range of the neutrino energy and by varying the mixing angles and the mass squared differences in the standard case. We found the cases where Independent Crossing Approximation breaks down for some unrealistic range of Δm_{31}^2 and θ_{13} . We also discuss the criterion which gives such independence.

1 Introduction

We now understand the solar neutrino problem [1], i.e., we now know that the deficit of the electron neutrino flux from the Sun can be accounted for approximately as the two flavor neutrino oscillations in matter. From the precise experiments, we know the mixing angle of the neutrino oscillations is large. It is called Large Mixing Angle solution. Historically, the so-called Small Mixing Angle solution has also been discussed [2], because it gave good fit to the solar neutrino data in the past. In this case, the nonadiabatic transitions, which are jumping between the different neutrino energy levels in matter, would become important. On the other hand, we also understand now the atmospheric neutrino problem [1], i.e., the deficit of the muon neutrino flux from the cosmic ray can be explained approximately as the two flavor neutrino oscillations. We also know the oscillation channel of the atmospheric neutrino differ from the solar neutrino. These lead us to conclude that the neutrino oscillation is described by three flavor neutrino oscillations.

In the case of the environments with high density, such as supernovae, it is important to consider the neutrino oscillations with the three flavor exactly, since there can be two energy level crossings. Furthermore, if nonadiabatic transitions occur at each level crossing at all, then we need to take them into account. In this case, there could be two nonadiabatic transitions [3]. The three flavor neutrino oscillations with two nonadiabatic transitions are very complicated in general. Although little is known about the exact solution of this very complicated problem, for instance, Ref. [4] assumes the Independent Crossing Approximation (ICA), which regards two nonadiabatic transitions as independent. In this approximation, we can regard each nonadiabatic transition locally as that between the two levels, and then we can get the total nonadiabatic transition just by multiplying each probability of the nonadiabatic transition [4]:

$$\hat{P}_c = \hat{P}_{\text{Lower}} \times \hat{P}_{\text{Higher}}$$

The reason that ICA has been assumed is (i) because the two energy ranges which give a resonance do not overlap for the solar and atmospheric neutrino oscillation parameters and (ii) because the energy of the supernovae neutrino is so low and the solar neutrino mixing angle is so large that we would not have nonadiabatic transitions at both of the energy level crossings.

However, if we have new physics beyond the standard model, then the standard matter effect may be modified [5, 6, 7] and it becomes unclear whether these reasonings always hold. The purpose of this paper is to exam-

ine whether ICA holds or not in the three flavor neutrino oscillations, and to discuss under which condition ICA breaks down. For simplicity we take only the three flavor case with the standard matter effect as an example and *we will assume hypothetical neutrino energy and hypothetical values of the oscillation parameters* to have nonadiabatic transitions at both energy level crossings.

From numerical calculations, we found that ICA breaks down for some range of Δm_{31}^2 and θ_{13} . We found that ICA breaks down when the ratio $\Delta m_{31}^2/\Delta m_{21}^2$ is of order one and the mixing angle θ_{13} is large. In the realistic range of the oscillation parameters, since $\Delta m_{31}^2/\Delta m_{21}^2$ is much larger than one and the mixing angle θ_{13} is small, the two neutrino energy ranges for the resonance never overlap. This is the reason why the treatment in Ref. [4] is regarded as correct. However, it turns out that it is not the correct criterion because we find that two level crossings become independent even if the two resonances overlap. Instead of using the notion of the overlapping resonances, we will introduce the new parameters which gives the criterion of ICA. With this criterion of ICA, we interpret the breaking of ICA as the sign of a large contribution of the extra off-diagonal coefficients. Our numerical calculations confirm this analytic treatment very well.

Although we discuss the simplest case only with the neutrino–electron interaction in this paper, we can apply our treatment to any other cases even if it is not known whether the nonadiabatic transitions are independent or not. For example, even if the effective ratio ($\Delta\tilde{m}_{31}^2/\Delta\tilde{m}_{21}^2$) in the presence of new physics is small, or the effective $\tilde{\theta}_{13}$ is large, we can discuss independence of the nonadiabatic transitions.

2 Neutrino oscillations with two flavor in matter

In this section, we review the standard treatment of the neutrino oscillations with two flavor.

2.1 Adiabatic transitions

The positive energy part of the Dirac equation for the flavor eigenstates $\nu_\alpha(t)$ ($\alpha = e, \mu$) propagating at time t in matter is given by

$$i\frac{d}{dt}\nu_\alpha(t) = H\nu_\alpha(t), \quad (1)$$

where H is given by

$$H = U \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix} U^{-1} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad (2)$$

here $\Delta \equiv \Delta m^2/2E$, $A \equiv \sqrt{2}G_F n_e(R)$ is the extra potential in medium at a distance $R = ct$ from the initial point, and U is the 2×2 MNS matrix [8]. H can be diagonalized as

$$H = \exp(i\sigma_2\tilde{\theta}) \text{diag}(\tilde{E}_1, \tilde{E}_2) \exp(-i\sigma_2\tilde{\theta}). \quad (3)$$

where σ_2 is the Pauli matrix and the effective mixing angle $\tilde{\theta}$ is given by

$$\sin^2 2\tilde{\theta} = \frac{(\Delta \sin 2\theta)^2}{(\Delta \cos 2\theta - A)^2 + (\Delta \sin 2\theta)^2}. \quad (4)$$

$\sin^2 2\tilde{\theta}$ becomes the maximum at the point $A(R_{\text{res}}) = \Delta \cos 2\theta$, which is called the Resonance point.

In the adiabatic case, the positive energy part of the Dirac equation for the mass eigenstates can be written as

$$i \frac{d}{dt} \begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \end{pmatrix} = \begin{pmatrix} \tilde{E}_1 & 0 \\ 0 & \tilde{E}_2 \end{pmatrix} \begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \end{pmatrix}, \quad (5)$$

where $\tilde{\nu}_j \equiv \exp(-i\sigma_2\tilde{\theta}) \nu_\alpha$ ($j = 1, 2$). Integrating Eq.(5), we obtain the survival probability of the electron neutrino

$$P(\nu_e \rightarrow \nu_e; R) = \cos^2 \left(\tilde{\theta}(R) - \tilde{\theta}(0) \right) - \sin \frac{\tilde{\theta}(R)}{2} \sin \frac{\tilde{\theta}(0)}{2} \sin^2 \int_0^R \frac{\Delta \tilde{E}}{2} dr, \quad (6)$$

where the difference of the eigenvalues is

$$\Delta \tilde{E} \equiv \tilde{E}_2 - \tilde{E}_1 = \sqrt{(\Delta \cos 2\theta - A)^2 + (\Delta \sin 2\theta)^2}. \quad (7)$$

In the case of the solar neutrino, since $\int_0^R \Delta \tilde{E} dr \gg 1$ and $|A(0)/\Delta| \gg 1$, we get

$$P(\nu_e \rightarrow \nu_e) = \sin^2 \theta. \quad (8)$$

2.2 Nonadiabatic transition

Let us now discuss Eq.(1) in the nonadiabatic case, i.e., in the case where we cannot ignore the variation of the effective mixing angle. In this case, we have

$$i \frac{d}{dt} \begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \end{pmatrix} = \begin{pmatrix} \tilde{E}_1 & -i\dot{\tilde{\theta}} \\ i\dot{\tilde{\theta}} & \tilde{E}_2 \end{pmatrix} \begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \end{pmatrix}. \quad (9)$$

The off-diagonal elements in Eq.(9) stand for an effect in which neutrino jumps the energy gap between \tilde{E}_1 and \tilde{E}_2 , and such an effect becomes non-negligible when the following the adiabatic condition breaks down:

$$\gamma \equiv \left| \frac{\Delta m^2}{2E} \frac{\sin^2 2\theta}{\cos 2\theta (\dot{n}_e/n_e)_{\text{res}}} \right| \gg 1, \quad (10)$$

which can be derived by comparing the diagonal and off-diagonal elements at the resonance point.¹

Zener [11] derived the jumping probability of spins in a linear magnetic field. In order for Zener's method to apply for neutrino, the density profile of electron has to be approximately linear. In that case, using the adiabatic condition Eq.(10), the jumping probability is given by

$$P_{\text{Zener}} = \exp\left(-\frac{\pi}{2}\gamma\right). \quad (11a)$$

Kuo and Pantaleone [12] showed that the differential equation for the exponential profile can be solved exactly. The analytical expression for the jumping probability is given by

$$P_{\text{exact}} = \frac{\exp\left[-\frac{\pi}{2}\gamma(1 - \tan^2\theta)\right] - \exp\left[-\frac{\pi}{2}\gamma\left(\frac{1 - \tan^2\theta}{\sin^2\theta}\right)\right]}{1 - \exp\left[-\frac{\pi}{2}\gamma\left(\frac{1 - \tan^2\theta}{\sin^2\theta}\right)\right]}. \quad (11b)$$

Using the jumping probability P_c , we can write down a simple expression for the survival probability of the electron neutrino. Again by taking the limits $\int_0^R \Delta\tilde{E}dr \gg 1$ and $|A(0)/\Delta| \gg 1$, we obtain

$$\begin{aligned} P(\nu_e \rightarrow \nu_e) &= \begin{pmatrix} \cos^2\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} 1 - P_c & P_c \\ P_c & 1 - P_c \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \sin^2\theta + P_c \cos 2\theta. \end{aligned} \quad (12)$$

The formulae of Eqs.(11) for the jumping probability differ when the mixing angle is large. In this case, especially for the extreme nonadiabatic transition, i.e., $\gamma \rightarrow 0$, the jumping probability P_c reaches the maximum value, i.e., complete conversion, in the Zener's solution, but it does not in the exact solution.

¹Here the resonance point differs from the Point of Maximal Violation of Adiabaticity [9, 10].

3 Neutrino oscillations with three flavor in matter

The positive energy part of the Dirac equation with three flavor in matter is given by

$$i\frac{d}{dt}\nu_\alpha(t) = H(t)\nu_\alpha(t), \quad (13a)$$

where $\alpha = e, \mu, \tau$, the Hamiltonian is

$$H(t) = U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 \\ 0 & 0 & \Delta_{31} \end{pmatrix} U^{-1} + \begin{pmatrix} A(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (13b)$$

and the standard parametrization [1] for the 3×3 MNS matrix is

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.$$

Here $s_{ij} \equiv \sin \theta_{ij}$, $c_{ij} \equiv \cos \theta_{ij}$, θ_{ij} are the mixing angles and δ is the CP phase as in the quark sector [13].

3.1 ICA with two nonadiabatic transitions

For the nonadiabatic transitions with three flavor, the neutrino can have two jumping points. When these two points are approximately far apart, from the analogy with Eq.(12), we get the survival probability of the electron neutrino [4]:

$$\begin{aligned} P(\nu_e \rightarrow \nu_e) &= (|U_{e1}|^2 \quad |U_{e2}|^2 \quad |U_{e3}|^2) \hat{P}_L \hat{P}_H \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= |U_{e1}|^2 P_L P_H + |U_{e2}|^2 P_H (1 - P_L) + |U_{e3}|^2 (1 - P_H), \end{aligned} \quad (14)$$

where

$$\begin{aligned} \hat{P}_L &= \begin{pmatrix} 1 - P_L & P_L & 0 \\ P_L & 1 - P_L & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \hat{P}_H &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - P_H & P_H \\ 0 & P_H & 1 - P_H \end{pmatrix}. \end{aligned}$$

Here, P_L and P_H are the jumping probabilities at the lower and higher crossing point each other, and U_{ei} are the elements of the three flavor MNS matrix.

3.2 The setting of our analysis

What we would like to discuss here is whether the ICA in Eq.(14) holds or not. Let us start our study by assuming several situations.

Throughout this paper, we will take the following reference values for the oscillation parameters and the reference function for the electron density $n_e(R)$ at distance R from the initial point:

$$\sin^2 2\theta_{12} = 0.87 \quad (15a)$$

$$\sin^2 2\theta_{23} = 1.0 \quad (15b)$$

$$\Delta m_{21}^2 = 7.9 \times 10^{-5} \text{ eV}^2 \quad (15c)$$

$$\delta = 0 \quad (15d)$$

$$n_e(R) \simeq 50 \times n_{e\odot}(R) \text{ cm}^{-3} \quad (15e)$$

$$n_{e\odot}(R) \simeq 245 N_A \exp(-10R/R_\odot) \text{ cm}^{-3}, \quad (15f)$$

where N_A is the Avogadro's number, $n_{e\odot}(R)$ stands for the electron density in the solar standard model [14], and R_\odot is the solar radius.

A few remarks are in order. (i) We refer to the standard value [1] for the first three oscillation parameters. (ii) We assume that the CP phase δ is equal to zero. This is because the transition probability of $\nu_e \rightarrow \nu_e$ (and its antineutrino) did not depend on the CP phase δ [15]. (iii) We assume the electron density which is proportional to the one in the Sun but is larger by a factor 50, in order to get the two energy level crossing.² For the exponential density profile, we can get a simple value $\dot{n}_e/n_e = \text{const.}$ for Eq.(10).

On the other hand, In order to have nonadiabatic transitions at the two level crossings, we will have to assume hypothetical values for the following parameters:

$$\begin{aligned} &\sin^2 2\theta_{31} \\ &\Delta m_{31}^2 \\ &E, \end{aligned}$$

where we will assume $\Delta m_{31}^2 > 0$, i.e., we will assume the so-called normal hierarchy, throughout this paper for ν_e (instead of $\bar{\nu}_e$) to have two level crossings.

Furthermore, we adopt the expression Eq.(11b) for P_L and P_H for the low (high). Here we take each of the adiabatic conditions, γ_L and γ_H as two

²If $|A/\Delta_{31}| \gg 1$, then it follows that $\tilde{\nu}_e(0) \simeq \tilde{\nu}_3(0)$

levels at each crossing point. From the analogy with Eq.(10) we have

$$\gamma_L \equiv \left| \frac{\Delta_{21} \sin^2 2\theta_{12}}{\cos 2\theta_{12} (\dot{n}_e/n_e)_{\text{res}}} \right| \quad (16a)$$

$$\gamma_H \equiv \left| \frac{\Delta \sin^2 2\theta_{13}}{\cos 2\theta_{13} (\dot{n}_e/n_e)_{\text{res}}} \right|, \quad (16b)$$

where we have defined,

$$\begin{aligned} \Delta_{21} &\equiv \Delta m_{21}^2/2E \\ \Delta_{31} &\equiv \Delta m_{31}^2/2E \\ \Delta &\equiv \Delta_{31} - \Delta_{21} \sin^2 \theta_{12}. \end{aligned} \quad (17)$$

In our study, we use the numerical calculation to check whether ICA holds or not. We use the Runge–Kutta method to solve numerically the positive energy part of the Dirac equation for three flavor (see Eqs.(13)).

3.3 Energy dependence of ICA

First of all, let us comment on the energy dependence of ICA. Since the Hamiltonian in vacuum is inversely proportional to the neutrino energy (see Eq.(13b)), the higher the neutrino energy is, the closer the distance of the two crossing points becomes. Figure 1 shows the asymptote of the energy diagram, in which the neutrino energy is 50MeV. Although the density profile is of the exponential type, the higher point is so near to the lower point that the density profile looks like linear. We would like to discuss this problem in more detail.

In Fig.2, we have plotted the probability $P(\nu_e \rightarrow \nu_e)$ by the analytic solution assuming ICA and that by the numerical one for each neutrino energy. The broken line (the box points) indicates the analytic (numerical) solution, respectively. The set of the oscillation parameters assumed here are

$$\begin{aligned} \sin^2 2\theta_{13} &= 0.08 \\ \Delta m_{31}^2 &= 2.4 \times 10^{-3} \text{ eV}^2. \end{aligned}$$

We assume the error of the numerical calculation within 0.005, and we find no difference between the two solutions in Fig.2. This result indicates that ICA holds even if the distance between two crossing points is close, i.e., the neutrino energy is high.

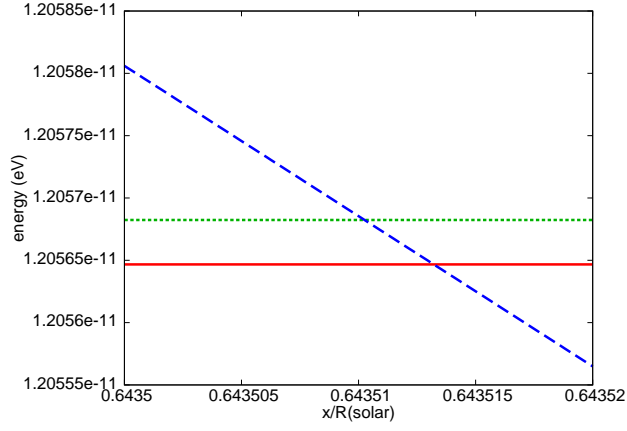


Figure 1: The diagonal elements of the neutrino in matter. Two crossing points are close.

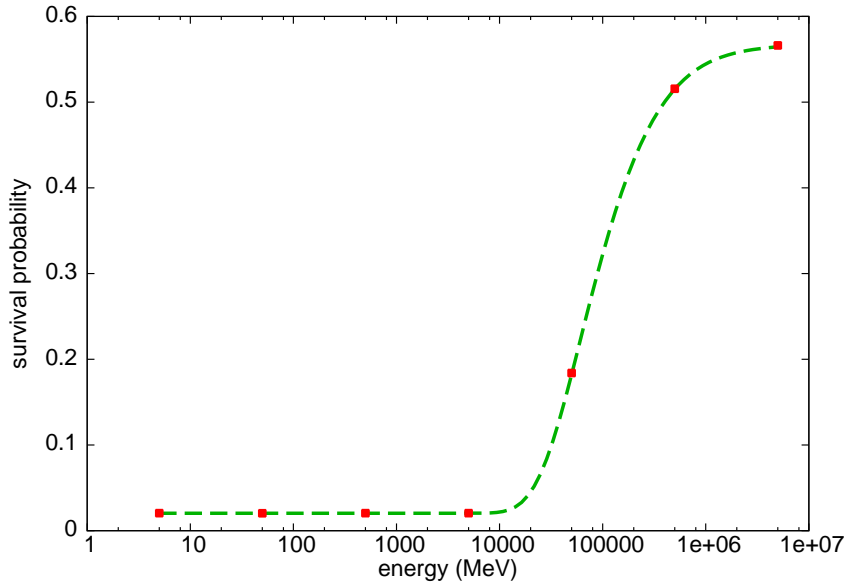


Figure 2: The difference of $P(\nu_e \rightarrow \nu_e)$ between by the analytic solution assuming ICA (the broken line) and by the numerical calculation (the box points). Here, $\sin^2 2\theta_{13} = 0.08$ and $\Delta m_{31}^2 = 2.4 \times 10^{-3} \text{ eV}^2$.

From the analogy with Eq.(4), we get

$$\sin^2 2\tilde{\theta}_{12} = \frac{(\Delta_{21} \sin 2\theta_{12})^2}{(\Delta_{21} \cos 2\theta_{12} - A)^2 + (\Delta_{21} \sin 2\theta_{12})^2} \quad (18a)$$

$$\sin^2 2\tilde{\theta}_{13} = \frac{(\Delta \sin 2\theta_{13})^2}{(\Delta \cos 2\theta_{13} - A)^2 + (\Delta \sin 2\theta_{13})^2}, \quad (18b)$$

where Δ has been defined in Eq.(17). In Fig.3, we have plotted the shapes of Eq.(18a)—the right line, and of Eq.(18b)—the left line, in the case:

$$\begin{aligned} \sin^2 2\theta_{13} &= 1 \times 10^{-6} \\ \Delta m_{31}^2 &= 2.4 \times 10^{-3} \text{ eV}^2 \\ E &= 700 \text{ MeV}, \end{aligned}$$

as the neutrino propagate in the medium to vacuum. Thus, two resonances seem to be far apart.

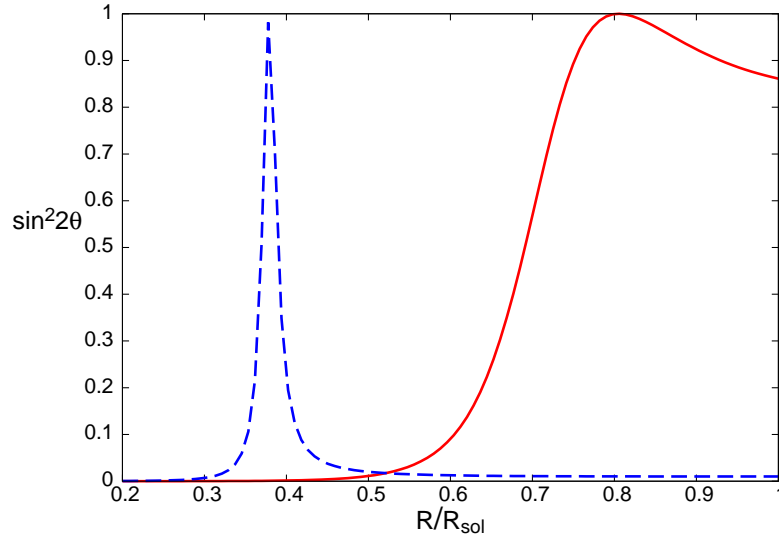


Figure 3: Each shape of two resonances, non-overlapping. The left line is the higher resonance—Eq.(18b). The right line is the lower resonance—Eq.(18a).

Introducing the notations

$$\begin{aligned} y &\equiv A/\Delta_{21} \\ \alpha &\equiv \Delta/\Delta_{21}. \end{aligned}$$

we can rewrite Eqs.(18),

$$\sin^2 2\tilde{\theta}_{12} = \frac{\sin^2 2\theta_{12}}{(\cos 2\theta_{12} - y)^2 + \sin^2 2\theta_{12}} \quad (19a)$$

$$\sin^2 2\tilde{\theta}_{13} = \frac{(\alpha \sin 2\theta_{13})^2}{(\alpha \cos 2\theta_{13} - y)^2 + (\alpha \sin 2\theta_{13})^2}. \quad (19b)$$

Thus, each resonance point is

$$y_H = \alpha \cos 2\theta_{13} \quad (20a)$$

$$y_L = \cos 2\theta_{12}, \quad (20b)$$

and each half width at half maximums is

$$\Gamma_H = \alpha \sin 2\theta_{13} \quad (21a)$$

$$\Gamma_L = \sin 2\theta_{12}. \quad (21b)$$

These quantities are independent of the neutrino energy. This means that two resonances never overlap even though the neutrino energy gets higher, i.e., even if the distance between two crossing points gets closer. From this one might be tempted to conclude that ICA always holds. The question we have to ask here is the validity of the non-overlapping resonance to judge whether ICA holds or not.

3.4 ICA for overlapping resonances

Let us now imagine the hypothetical situation:

$$\Delta m_{31}^2 \rightarrow 0.04 \times \Delta m_{31}^2 \sim \Delta m_{21}^2,$$

that is,

$$\Delta m_{31}^2/\Delta m_{21}^2 \simeq O(1).$$

This parameter realizes the overlapping resonances (see Fig.4). Or, let us here imagine another hypothetical situation,

$$\sin^2 2\theta_{13} = 0.75 (\theta_{13} = 30^\circ).$$

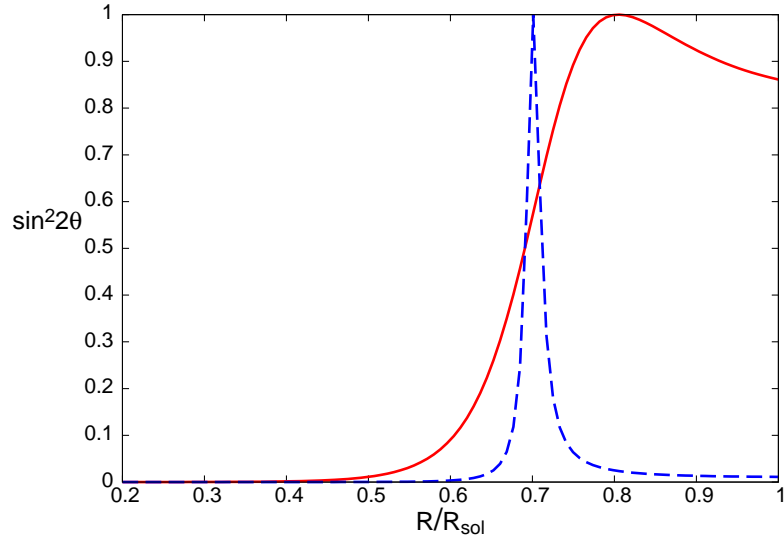


Figure 4: The modified resonances—overlapping. Here, $\Delta m_{31}^2/\Delta m_{21}^2 \simeq O(1)$.

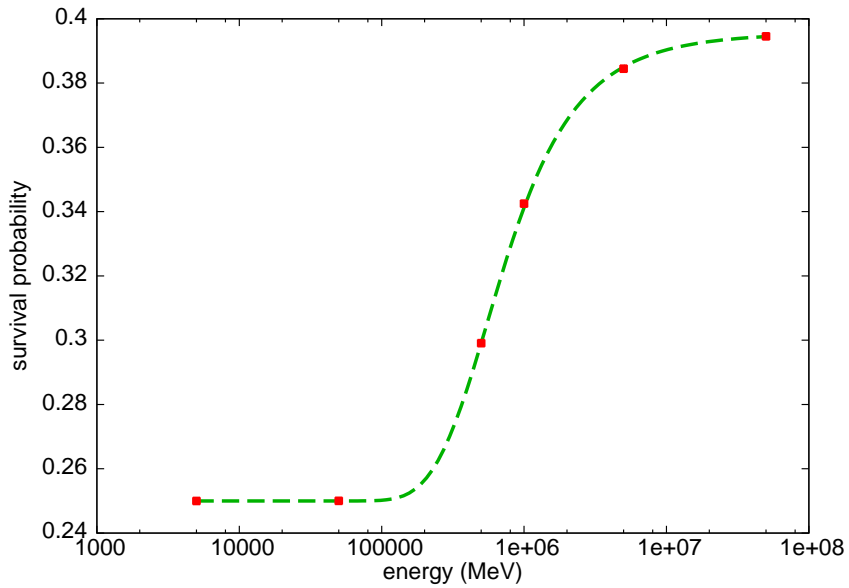


Figure 5: The difference of $P(\nu_e \rightarrow \nu_e)$ between by the analytic solution assuming ICA (the broken line) and by the numerical calculation (the box points). Here, $\sin^2 2\theta_{13} = 0.75$ and $\Delta m_{31}^2 = 2.4 \times 10^{-3} \text{eV}^2$.

This parameter realizes the overlapping resonance too.

In Fig.5, we have plotted the both probabilities $P(\nu_e \rightarrow \nu_e)$ for the second situation ($\sin^2 2\theta_{13} = 0.75$). Again we find no difference between the two solutions. From this we observe that overlapping of the two resonances is not a sufficient condition for ICA to break down.

3.5 Dependence of ICA on the mixing angle and the mass squared difference

To search for the parameter range of ICA, we have examined several cases for θ_{13} and Δm_{31}^2 , and tested whether ICA held or not. For example, in the case:

$$\begin{aligned}\Delta m_{31}^2/\Delta m_{21}^2 &\simeq O(1) \\ \sin^2 2\theta_{13} &= 0.08,\end{aligned}$$

we found that the ICA breaks down (see Fig.6).

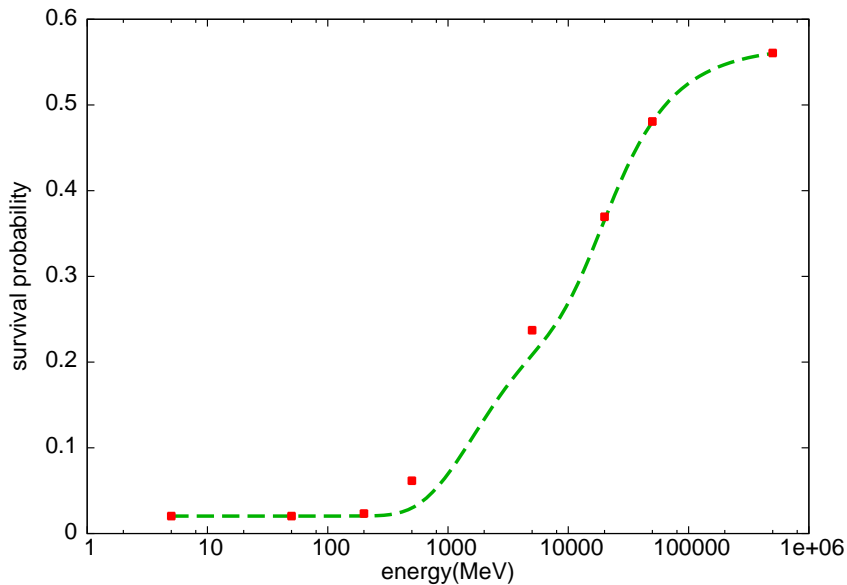


Figure 6: The ICA breaking ($\Delta m_{31}^2/\Delta m_{21}^2 \simeq O(1)$, $\sin^2 2\theta_{13} = 0.08$).

Table 1: The difference of $P(\nu_e \rightarrow \nu_e)$ between by the analytic solution assuming ICA and by the numerical calculation. Nonzero terms mean that ICA is broken.

$\sin^2 2\theta_{13}$	$\Delta m_{13}^2/\Delta m_{12}^2$	1.2	3.0	15	30
0.75		0.006	0.000	0.000	0.000
0.08		0.048	0.000	—	0.000
10^{-6}		0.000	—	—	0.000

In Tab.1, we have shown the difference between $P(\nu_e \rightarrow \nu_e)$ by the analytic solution assuming ICA and the one by the numerical one for several sets of $\sin^2 2\theta_{13}$ and $\Delta m_{31}^2/\Delta m_{21}^2$. The numerical value 0.000 indicates that the difference is less than 0.005, the error of the numerical calculations. In such cases, we interpreted that ICA holds. The result implies that ICA is not applicable when θ_{13} is large and when $\Delta m_{31}^2/\Delta m_{21}^2$ is small.³

3.6 The criterion of ICA

Let us again consider the positive energy part of the Dirac equation for the neutrino in medium. From Eq.(13b), we obtain

$$i\frac{d}{dt}\tilde{\nu}_i = \tilde{U}^{-1}H_f\tilde{U}\tilde{\nu}_i - i\tilde{U}^{-1}\frac{d\tilde{U}}{dt}\tilde{\nu}_i. \quad (22)$$

In the adiabatic case, we can ignore the second term on the right hand side of Eq.(22). Doing this matrix arithmetic, we have

$$i\frac{d}{dt}\begin{pmatrix} \tilde{\nu}_1(t) \\ \tilde{\nu}_2(t) \\ \tilde{\nu}_3(t) \end{pmatrix} = \begin{pmatrix} \tilde{E}_1 & -i\dot{\tilde{\theta}}_{12} & -i\tilde{c}_{12}\dot{\tilde{\theta}}_{13} \\ i\dot{\tilde{\theta}}_{12} & \tilde{E}_2 & i\tilde{s}_{12}\dot{\tilde{\theta}}_{13} \\ i\tilde{c}_{12}\dot{\tilde{\theta}}_{13} & -i\tilde{s}_{12}\dot{\tilde{\theta}}_{13} & \tilde{E}_3 \end{pmatrix} \begin{pmatrix} \tilde{\nu}_1(t) \\ \tilde{\nu}_2(t) \\ \tilde{\nu}_3(t) \end{pmatrix}. \quad (23)$$

³This is equal to the condition for which the two flavor approximation fails.

In order for ICA to apply to Eq.(23), the Hamiltonian H_L at the lower resonance has to be:

$$H_L = \begin{pmatrix} \tilde{E}_1 & -i\dot{\tilde{\theta}}_{12} & 0 \\ i\dot{\tilde{\theta}}_{12} & \tilde{E}_2 & 0 \\ 0 & 0 & \tilde{E}_3 \end{pmatrix}. \quad (24a)$$

Likewise, we have to have the following Hamiltonian H_H at the higher resonance, for ICA to apply:

$$H_H = \begin{pmatrix} \tilde{E}_1 & 0 & 0 \\ 0 & \tilde{E}_2 & i\dot{\tilde{\theta}}_{13} \\ 0 & -i\dot{\tilde{\theta}}_{13} & \tilde{E}_3 \end{pmatrix}. \quad (24b)$$

In this case, we can get Eq.(14). From this, we observe that the ICA breaks down, for example, when the extra terms depending on $d\tilde{\theta}_{13}/dt$ becomes large in H_L at the lower resonance.

Let us introduce the new parameter for the indicator of the ICA:

$$\kappa_L \equiv \left(\frac{d\tilde{\theta}_{13}}{dt} / \Delta \tilde{E}_{21} \right)_L. \quad (25)$$

When this parameter is large, it means that the contribution of $d\tilde{\theta}_{13}/dt$ is large in H_L .

In Fig.7, where the x-axis (y-axis) is $\Delta m_{31}^2/\Delta m_{21}^2$ ($\sin^2 2\theta_{13}$), we have drawn the contour lines of κ_L with the neutrino energy satisfying $\gamma_L = 1$ (see Eq.(10)), for $\kappa_L = 1.0, 0.5, 0.1$. Thus, the new parameter κ_L describes the results in Tab.1 very well.

We can rewrite κ_L^{-1} as

$$\begin{aligned} \kappa_L^{-1} &= \Delta_{21} \sin 2\theta_{12} / \left(\frac{\dot{A} \sin^2 2\tilde{\theta}_{13}}{\Delta \sin 2\theta_{13}} \right)_L \\ &= \frac{\Delta \sin 2\theta_{12}}{\cos 2\theta_{12} (n_e/n_e)_L} \times \frac{(\cos 2\theta_{13} - (\Delta_{21}/\Delta) \cos 2\theta_{12})^2 + \sin^2 2\theta_{13}}{\sin 2\theta_{13}}. \end{aligned}$$

Fixing the neutrino energy by the condition $\gamma_L = 1$, we get

$$\kappa_L^{-1}(E; \gamma_L = 1) = \left(\frac{\Delta}{\Delta_{21}} \right) \frac{(\cos 2\theta_{13} - (\Delta_{21}/\Delta) \cos 2\theta_{12})^2 + \sin^2 2\theta_{13}}{\sin 2\theta_{12} \sin 2\theta_{13}}. \quad (26a)$$

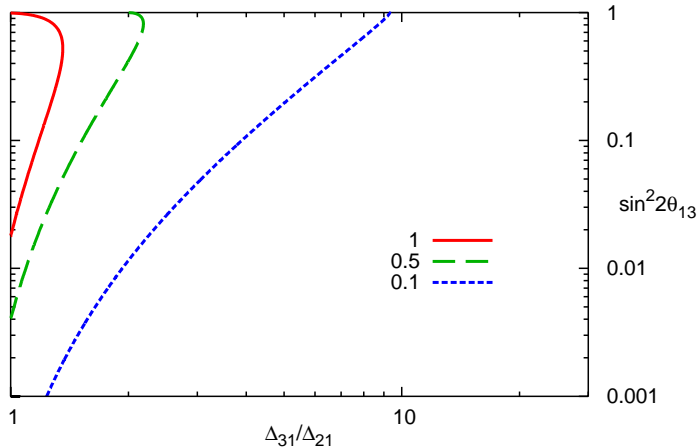


Figure 7: The contour map of κ_L .

As in Eq.(26a), we can get κ_H^{-1} at the higher resonance

$$\kappa_H^{-1}(E; \gamma_H = 1) = \left(\frac{\Delta_{21}}{\Delta} \right) \frac{((\Delta/\Delta_{21}) \cos 2\theta_{13} - \cos 2\theta_{12})^2 + \sin^2 2\theta_{12}}{\sin 2\theta_{12} \sin 2\theta_{13}}. \quad (26b)$$

Eqs.(26) are nothing but the condition of whether ICA holds or not, i.e., the condition for ICA as functions of θ_{13} and $\Delta m_{31}^2/\Delta m_{21}^2$. It is remarkable that these parameters in Eqs.(26) are independent of the density profile $n_e(R)$.

4 Discussion

In the previous section, we have investigated whether ICA held or not, and the neutrino energy which requires ICA to break down ranges in the interval $5\text{MeV} \leq E \leq 50\text{TeV}$. At high energy, the effect of the absorption of the neutrino by medium becomes nonnegligible. Let us consider this effect first. The reaction number of the neutrino N (/sec) is

$$\begin{aligned} N &= L \times \sigma(\nu N) \\ &= N_A \times \rho \times \sigma \times R \\ \sigma(\nu N) &= 0.68 \times 10^{-38} \times E \text{ (cm}^2\text{)}, \end{aligned}$$

where L is the luminosity of neutrino, $\sigma(\nu N)$ is the cross section of neutrino–nucleus and ρ is the matter density. Here we estimated the matter density $\rho = n_e(R)$ at $n_e(0) = \text{const}$. Thus, the flight ranges at each neutrino energy

for the present density profile Eq.(15e) are

$$\begin{aligned} R &\simeq 2 \times 10^{-1} \text{ km} \quad (1\text{PeV}) \\ &\simeq 2 \times 10^2 \text{ km} \quad (1\text{TeV}) \\ &\simeq 2 \times 10^5 \text{ km} \quad (1\text{GeV}). \end{aligned}$$

Although this is a rough estimate, we find that the neutrino, whose energy is not less than 1TeV, cannot reach the vacuum area. Therefore, our hypothesis that high energy neutrinos are emitted at the center of a supernova and are observed outside of the supernova may not make sense.

Secondly, in dense matter for which there are two crossing points, the neutrino density $n_\nu(R)$ becomes large too. In such a case, the neutrino–neutrino interaction is strong [16, 17, 18, 19, 20, 21, 22]. Adding this effect, we need to incorporate $n_\nu(R)$, which is time–dependent, into the off–diagonal parts for H (see Eqs.(13)). No formula of the nonadiabatic transitions attended by this effect have been established even the formulae with two flavor. In our study, we have assumed that the two jumping points are far apart from the range where the neutrino–neutrino interactions become important.

5 Conclusion

In this paper, we addressed the question whether the Independent Crossing Approximation (ICA) holds or not. Ref. [4] has focused on the picture of the non–overlapping resonance by which they judge whether two nonadiabatic transitions are independent or not. One of the purposes of this study is to check whether this interpretation is correct or not.

First of all, we have checked the dependence of ICA on the neutrino energy. Namely, we checked the dependence of ICA on the distance between the higher and lower crossing points. By numerical calculations, we showed that ICA is independent of the neutrino energy, that is, the distance between two crossing points.

Secondly, we have checked the validity of the picture of the non–overlapping resonance for ICA by varying the parameters θ_{13} and Δm_{31}^2 in the hypothetical ranges. We found numerically that ICA can hold even if two resonances overlap, i.e., ICA does not always break down even if two resonant widths overlap.

Thirdly, we have searched for the case in which ICA breaks down. We found that ICA breaks down when θ_{13} is large and when $\Delta m_{31}^2/\Delta m_{21}^2$ is small.

Finally, we have introduced the new parameters as the criterion of ICA. With this criterion of ICA, we interpret the ICA breaking as the sign of a large

contribution of the off-diagonal coefficients, for example, the contribution of $d\tilde{\theta}_{13}/dt$ at the lower resonance. We have shown that the new parameters—the criterion of ICA, taking the extra contribution into consideration, describe well whether ICA holds or not.

From the recent experiments, we know that the ratio $\Delta m_{31}^2/\Delta m_{21}^2$ is large [1] and the mixing angle θ_{13} is small [23]. Therefore the ICA breaking is tiny enough for the two nonadiabatic transitions to be regarded as independent, and all we need is just to multiply \hat{P}_H and \hat{P}_L which are obtained by the two flavor formalism, as in Ref. [4].

In this paper, we have dealt with the only the neutrino–electron interaction. In other cases, for example, where neutrino transition moments couple to the large magnetic fields [25, 26, 27, 28], under the influence of the neutrino–neutrino interaction [16, 17, 18, 19, 20, 21, 22], or under the influence of the Non-Standard Interaction [5, 6, 7], our procedure is necessary.

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