Theory in Non(anti)commutative Superspace and Twisted symmetry

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Abstract

We investigate a non(anti)commutative $\mathcal{N} = 1$ superspace with a Hopf algebraic method. We review briefly noncommutative theories in both spacetime and superspace in usual procedures, by the Weyl mapping and Moyal product. Especially an unfamiliar noncommutativity between a spacetime coordinate and a fermionic superspace coordinate is reviewed further. After these reviews, we describe the foundations of our method, including a brief review of Hopf algebra and Drinfel'd twist, the construction of Twisted Poincaré algebra as a example. We study the extension of the Hopf algebraic procedure to supersymmetric theory. Twisted Super Poincaré algebra is constructed and non(anti)commutative superspace appears as the representation of that. The noncommutative deformation is performed to maintain twisted symmetries, namely twisted Lorentz and twisted supersymmetric invariant, although the deformed theory is the same as what is formulated with usual Moyal product. Under this procedure the algebraic structure of the symmetry is not deformed, while the multiplication rule on the representation space is modified. We also consider twisted superconformal algebra, and find that various exotic noncommutative relations between coordinates appear in superspace. We show how to give the Fuzzy-sphere-like noncommutativity with this procedure, too.

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1 Introduction

1.1 Noncommutative Theory

We will consider the formulation and symmetry of noncommutative theories in this thesis. Noncommutative theory which we will discuss here is the theory on a spacetime with noncommutative coordinates.[1].

To begin with, we overview a historical summary and the current developments of noncommutative theory.

The concept of noncommutative coordinate is an old idea in particle physics. The noncommutative property of physical quantities, in particular the position and momentum of a particle, is a central idea of *quantum mechanics*. Then it is no wonder that people at the dawn of quantum mechanics looked for the possibility that spacetime coordinates may not commute with each other, just like a coordinate does not commute with its canonical conjugate momentum. Noncommutative theory has a long history which goes back at least nearly sixty years. It was Snyder who first published on the noncommutative theory in concrete terms [1][2]. He introduced a parameter of length and modify the commutators between spacetime coordinates with the parameter to connect with the generators of Lorentz group. That length gives the smallest cell size of space-time in Lorentz invariant way, and he argued that the noncommutativity might control a divergence of a quantum calculation, which had bothered physicists at that time. However, it did not attract much interest because of the development of *renormalization*.

Nowadays noncommutative theory becomes common in theoretical physics broadly. It appears even in solid physics. The dynamics of a particle under a constant magnetic field, in strong limit, can be formulated by the noncommutative canonical coordinate in quantum mechanics. It is a key technology of a description of quantum Hall effect.

It has also been investigated in mathematical aspects. *Noncommutative geometry* is a major

[1] We assume the word *noncommutative* is used only in this meaning in the thesis to prevent confusion between noncommutative and non-Abelian.

[2] Heisenberg is the first person to mention the possibility of noncommutative coordinate[1].

subject in modern mathematics under intensive research, and it turned out to have much fruitful relationship with physics. See, e.g. [2].

Recently noncommutative theory draws our interest in relation to superstring theory. In principle, string theory has a smallest building block of spacetime. A string is an object which is not point-like but is of finite size comparable to the Planck length in a target spacetime, thus that may look to us as though the coordinates has a uncertainty. Several researches postulate a spacetime uncertainty relation[3],

$$\Delta x^{\mu} \Delta x^{\nu} \gtrsim \ell_p^2, \tag{1.1}$$

here ℓ_p^2 is the Planck length of the target space. That means essentially *the quantization of spacetime*, and causes the noncommutative coordinates in a spirit of uncertainty relation.

It is true that string theory involves noncommutative spacetime in its own right. On the other hand, in resent years D-brane scenario shed light on the noncommutative spacetime which arises from string theory. Seiberg and Witten[7] pointed out that if there is non-zero NS(Neveu-Schwarz)-NS antisymmetric tensor B-field background, then coordinates on D-brane become noncommutative in the zero slope limit $\alpha \to 0$,

$$[x^{\mu}, x^{\nu}] = i\Theta^{\mu\nu} \neq 0, \tag{1.2}$$

where $\Theta^{\mu\nu}$ is a constant with anti-symmetric indices. The noncommutativity parameter $\Theta^{\mu\nu}$ corresponds to the vacuum expectation value (VEV) of background B-field. The VEVs depend on the dynamics behind the theory, therefore the scale of non-commutativity is more arbitrary than what naturally arises from string. Naively we assume that they are at the Planck scale since they come from gravitational quantum corrections. A low energy effective theory which we get in this case is *noncommutative Yang-Mills theory* on the spacetime with the noncommutative relation(1.2).

String theory is the most promising candidate of a unified theory of physics, or *the theory of everything*, including a quantum theory of gravity. In addition, almost every notable candidate of a fundamental theory in high energy physics, such as matrix model, loop gravity, etc., predicts, more or less, the appearance of noncommutative geometry. So these facts strongly indicate the possibility that the true description of our world may be non-commutative field theory in four-dimensional space-time in some energy region.

1.2 Symmetry Breaking in Noncommutative Spacetime

Although noncommutative theory is derived from string theory, it is difficult to investigate. In practice we study a noncommutative theory in the framework of the well-established quantum field theory (QFT), through the deformation of the noncommutative space. If we have reasonable evidences for an appearance of noncommutative geometry based on a reliable high energy theory, it is natural to ask what happens when spacetime noncommutativity is introduced into the well-known theories, say the standard model.

However there is one problem. If we treat a noncommutative theory within QFT language, then we have to introduce noncommutativity parameters, which are often dimensionful, into the theory by hand. To make matters worse, in general it breaks the symmetries which the theory originally has. For example, it is well known that a noncommutative relation (1.2) breaks Lorentz symmetry.

This fact is in distinct contrast to the noncommutative theory which appears from some theory in higher energy, like string theory. In such case, even if the symmetry looks like to be broken, it is caused by having of nonzero (or asymmetric point) VEVs of certain fields dynamically, not by hand. The higher theory is still regarded as fully symmetric, and formulated in a covariant way. Dynamical symmetry breaking is well known scenario, *e.g.* the Higgs mechanism, and the breaking is only fake one. Basically it is restored in a high energy region.

The difference between two cases is summarized in Table 1.

String theory	Noncommutative QFT
• Spontaneous symmetry breaking.	• Explicit symmetry breaking.
• Recoverable breaking from the dynamics.	• Unavoidable breaking by introducing (dimensionful) noncommutativity parameters by hand.
• Formulated in a fully symmetric and covariant way.	• Symmetric theory + perturbation induced by small symmetry breaking.

Table 1: Comparison of the symmetry breaking in noncommutative theory

1.3 Observer and Particle Lorentz Transformation

The noncommutative relation Eq.(1.2) breaks Lorentz symmetry as previously mentioned. Lorentz symmetry is one of the most important symmetry in physics. We have not had any experimental evidence of the violation of the symmetry, even though it has been tested intensively in many ways. If there is no Lorentz symmetry, we immediately face difficulties.

What is the Lorentz symmetry breaking in the noncommutative theory? In this section we confine our attention to Lorentz symmetry and its breaking, in the noncommutative spacetime with the relation Eq.(1.2). Since constant noncommutativity parameters $\Theta^{\mu\nu}$ have the indices of spacetime, which can be regarded as a constant background tensor-like field. In fact it appears in string theory.

In this case, we can consider two kinds of Lorentz transformation (Fig. 1). First is the *Observer Lorentz Transformation*. The observer Lorentz transformation is the transformation of a movable observer, in which only the coordinate of the observer transforms while all other objects stay. Viewed from the opposite side, it is equivalent to say that whole spacetime transforms in the opposite direction except the observer. Thus the observer Lorentz transformation is related to a global transformation. It means that constant parameters $\Theta^{\mu\nu}$ also transform like tensors. If $\Theta^{\mu\nu}$ transform like tensors, the dynamics is not changed, because the terms which contains $\Theta^{\mu\nu}$ contracted with other fields in the Lagrangian remain Lorentz invariant. No physics is changed in the process, therefore the observer inevitably observes the same behavior of an event.

The second transformation is *Particle Lorentz Transformation.* That is the transformation of not an observer but an object in physics. If a particle changes direction or is boosted, then it changes its relative state in the background fields, namely relative direction or speed. This transformation is related to a local Lorentz transformation, then $\Theta^{\mu\nu}$ *do not transform* in the process, they behave precisely like constant. The Lagrangian is no longer invariant if it contains the interaction of the particle with spacetime indices contracted with $\Theta^{\mu\nu}$.

In ordinary commutative space, these two Lorentz symmetries are equivalent. Lagrangian is constructed with local fields and Lorentz invariant constants. No constant depends on a direction in the spacetime, so Lorentz symmetry is manifest, at least at the classical level.

However this equivalence no longer holds in the noncommutative spacetime. The noncommutative relation Eq.(1.2) preserve *Observer Lorentz symmetry*, but breaks *Particle Lorentz symmetry.*

1.4 Noncommutative Space as a Representation of Hopf Algebra

Certain theories predict the noncommutativity. However they are too complicated for the purpose of an effective study of noncommutative theory in general. Above all things, we do not even know which theory describe our physics correctly. So we want to treat the noncommutative theory apart from the higher theory from a practical point of view. But treating a noncommutative theory within standard QFT, we have to introduce noncommutativity parameters into the theory, and we will face the problem of symmetry breaking.

Recently an idea to improve the situation is suggested [5, 6]. Their strategy is a realization of a noncommutative space (1.2) as the representation of a deformed Poincaré algebra. In ordinary theories, commutative space is a representation of some symmetry algebra. As we deform the symmetry algebra, its representation is also deformed to correspond with it (Fig.2).

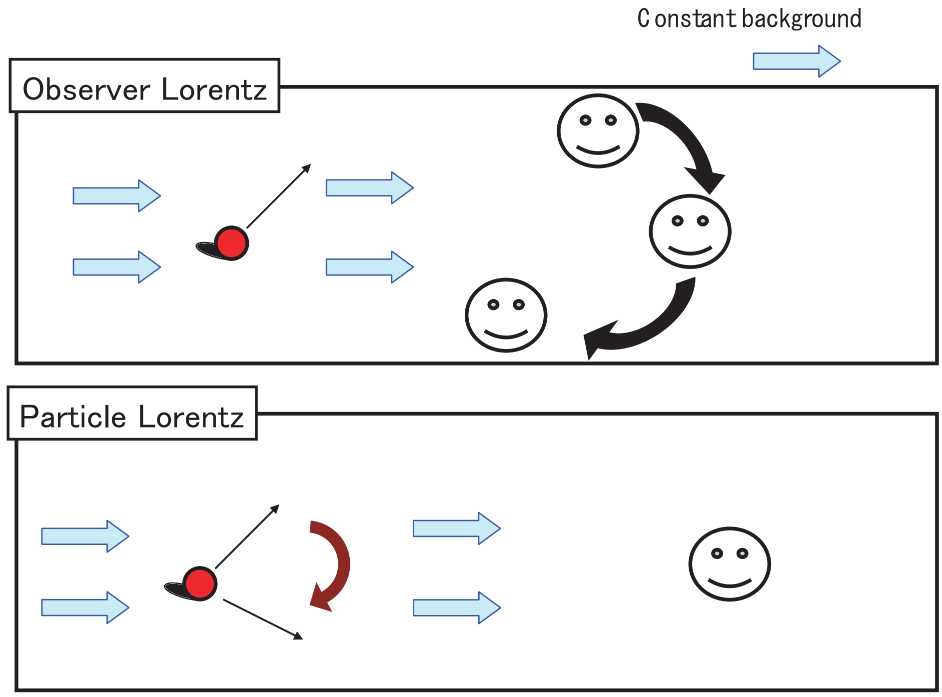

Figure 1: Observer Lorentz and particle Lorentz. The upper figure shows the observer Lorentz transformation. The observer watches the same phenomena everywhere he is. The lower figure shows the particle Lorentz transformation. The relative direction in the background field is changed, therefore the observer watches different phenomena.

Chaichian *et. al* claimed that the original symmetry of a theory is broken indeed by introducing noncommutativity, but the deformed symmetry can remain.

Our work [31] is essentially an extension of their work to a supersymmetric case. We can construct various non(anti)commutative superspace by the above procedure, to maintain the deformed symmetries. This is a main topic in this thesis.

$$\begin{tikzcd}
\boxed{\textbf{Symmetry algebra}} \arrow[r, Leftrightarrow] \arrow[d, "\text{Drinfel'd twist}"] & \boxed{\textbf{Commutative space}} \arrow[d] \\
\boxed{\textbf{Twisted Hopf algebra}} \arrow[r, Leftrightarrow] & \boxed{\textbf{Noncommutative space}}
\end{tikzcd}$$

Figure 2: Noncommutative space is the representation of a deformed symmetry.

The organization of this paper is as follows. In the next section we give a brief introduction of a formulation of noncommutative theory, which is the standard procedure with a Moyal-product both in noncommutative space and superspace. In the section 3, we introduce our study of a novel noncommutative superspace, that is the noncommutativity between spacetime coordinate and fermionic coordinate in superspace. The section 3 deviate slightly from the main subject in this thesis. But our work[25] is the first rigorous attempt to investigate such type of noncommutative theory including its quantum properties, we have decided to make a independent section. The section 4 is devoted the main idea and necessary tools for the formulation in Hopf algebra. We construct a twisted Poincaré algebra, and show that the canonical type noncommutative space (1.2) appears as the representation space of the twisted Poincaré algebra. In the section 5, we describe how to extend the procedure to the supersymmetric theory. This is a major contribution of our work. In addition, we show other twisted symmetries and the induced non(anti)commutative superspace. The various exotic noncommutativity which is not yet known is realized. In the last section we summarize our works, and discuss some prospective future investigations.

2 Noncommutative Theory

In this section, we review a standard formulation of noncommutative theory, in both ordinary spacetime and superspace, for comparison with our approach in the later sections.

2.1 Spacetime noncommutativity

We are already familiar with a procedure of introducing noncommutativity, that is the canonical quantization in quantum mechanics. Noncommutativity between spacetime coordinates is introduced along the similar lines. Generally speaking, as in the case for quantum mechanics, we replace the spacetime coordinates x^μ with the corresponding operators $\hat{x}^\mu$ on Hilbert space and impose certain commutator relations on them.

Noncommutative theory could be viewed as the theory in unital algebra $\mathcal{A}_\Theta$. $\mathcal{A}_\Theta$ is a linear space and generated by 1(unit) and x^μ, on which the relation $[x^\mu, x^\nu] = i\Theta^{\mu\nu}$ is realized. $\Theta^{\mu\nu}$ is a deformation parameter on $\mathcal{A}_\Theta$, and when $\Theta^{\mu\nu}$ goes to zero, we recover the usual commutative description of spacetime. That means *the existence of the commutative limit* of the theory. In the case of quantum mechanics, the quantum theory has been deformed with the Planck "parameter" $\hbar$, and in the classical limit $\hbar \to 0$ the theory reduces to the corresponding classical theory. The noncommutative theory has been deformed with the noncommutativity parameters $\Theta^{\mu\nu}$, and in the commutative limit $\Theta^{\mu\nu} \to 0$ the theory reduces to the corresponding theory on commutative space. We can discuss the two procedures in parallel.

Let us start with a fairly general situation. The commutation relation between spacetime coordinates can be written with the noncommutativity parameters which is a function on spacetime. In an operator description, the commutator may be written with a function of the operators.

$$[\hat{x}^\mu, \hat{x}^\nu] = \Theta^{\mu\nu}(\hat{x}), \tag{2.1}$$

where $\Theta^{\mu\nu}$ is a function which has the dimension of (length)^2, with antisymmetric indices and can be expanded in a formal Taylor series,

$$\Theta^{\mu\nu}(\hat{x}) = \Theta^{\mu\nu}_{(0)} + \Theta^{\mu\nu}_{(1)\rho}\hat{x}^\rho + \Theta^{\mu\nu}_{(2)\rho\sigma}\hat{x}^\rho\hat{x}^\sigma + \cdots . \tag{2.2}$$

Here $\Theta^{\mu\nu}_{(k)\rho_1\cdots\rho_k}$ are constant coefficients of order k with a dimension $(\text{length})^{2-k}$. Note that only a set of $k = 2$ coefficients $\Theta^{\mu\nu}_{(2)\rho\sigma}$ is dimensionless. If the scale of the noncommutativity parameter as a function on spacetime is sufficiently small, *i.e.*, the noncommutativity is ruled by informations in the immediate vicinity, we can ignore the terms of higher order of x and it can be approximated by the first few terms in the sense of a Taylor expansion.

The first constant term in the expansion Eq.(2.2) leads to

$$[\hat{x}^\mu, \hat{x}^\nu] = \Theta^{\mu\nu}_{(0)}. \tag{2.3}$$

It is called *canonical type* noncommutativity. The type of this noncommutativity has already appeared in Eq.(1.2) and is most intensively investigated.

For $k = 1$, we have the simplest non-constant noncommutative deformation, called noncommutativity of the *Lie algebra type*.

$$[\hat{x}^\mu, \hat{x}^\nu] = \Theta^{\mu\nu}_{(1)\rho}\hat{x}^\rho \tag{2.4}$$

It appears, for example, in κ-Minkowski spacetime[8].

In next order we rewrite the relation

$$\hat{x}^\mu\hat{x}^\nu = \frac{1}{q}R^{\mu\nu}{}_{\rho\sigma}\hat{x}^\rho\hat{x}^\sigma, \tag{2.5}$$

where $R^{\mu\nu}{}_{\rho\sigma}/q = \Theta^{\mu\nu}_{(2)\rho\sigma} + \delta^\mu{}_\rho\delta^\nu{}_\sigma$. This noncommutativity is the *q-deformation type* in the quantum group sense.

We focus on *canonical* type deformation in the following. More general noncommutativity has been considered, for example in [11].

2.2 Weyl Mapping and Moyal Product

Weyl Quantization in quantum mechanics replaces a position q and momentum p, which are arguments of a function in phase space, with corresponding operators $\hat{q}$ and $\hat{p}$ respectively.

$$F(p,q) \to \frac{1}{(2\pi)^2}\int d\eta d\xi \tilde{F}(\eta,\xi)\exp\left(i\hat{p}\eta + i\hat{q}\xi\right) \tag{2.6}$$

$\tilde{F}(\eta,\xi)$ is a Fourier function of $F(p,q)$.

$$\tilde{F}(\eta,\xi) = \int dpdqF(p,q)\exp\left(-ip\eta - iq\xi)\right) \tag{2.7}$$

Using Eq.(2.6), we associate the functions in the Schrödinger formalism with the corresponding operators on a Hilbert space in the operator formalism.

In this section, Weyl quantization of noncommutative spacetime is reviewed based on the review by Szabo[9]. We consider the canonical noncommutativity, so noncommutativity parameter $\Theta^{\mu\nu}$ is constant.

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}. \tag{2.8}$$

Noncommutativity of time coordinate causes many physical problems. First of all, we could hardly relate noncommutative "time" to our time which describes when the physical events occurs. Moreover unitarity of the system becomes not clear. Even if only spacelike noncommutativity is imposed at first, timelike noncommutativity appears when Lorentz transformation is performed. For details of noncommutativity and unitarity, see for example, [12].

An easy way to avoid these difficulties is that we forget about the time coordinate temporarily and work in D-dimensional Euclid space $\mathbb{R}^D$.

In a rigorous mathematical treatment, we have to confirm the convergence of the calculations. To guarantee the existence of the well-defined Fourier transformed function $\tilde{F}$, the function on spacetime F and the derivative of F at any order should fall off quickly enough at infinity to converge the integral and to avoid effects from the boundary. It requires the fall off is stronger than any power-law, *i.e.*

$$\sup_x \left(1+|x|^2\right)^{k+n_1+n_2+\cdots+n_D} |\partial_1^{n_1}\partial_2^{n_2}\cdots\partial_D^{n_D}F(x)|^2 < \infty \tag{2.9}$$

where k,n_i are certain positive integers. This condition guarantees the well-defined algebraic structure of the differential function of F at all orders. Functions on spacetime F span Banach space with L^∞-norm,

$$||F||_\infty = \sup_x |F(x)|. \tag{2.10}$$

Through that, it enables us to change the physical description from functions on commutative differential manifold into coordinate space representation on Hilbert space.

We define the following (operator) integral kernel,

$$\hat{\Delta}(x) = \int \frac{d^D k}{(2\pi)} e^{ik_i\hat{x}^i} e^{-k_i x^i}. \tag{2.11}$$

$\hat{\Delta}(x)$ is a Hermitian operator $\hat{\Delta}(x) = \hat{\Delta}(x)^\dagger$, and in the commutative limit, *i.e.* $\Theta^{\mu\nu} = 0$, it is reduced to the delta function $\delta^D(\hat{x}-x)$. This integral kernel gives the *Weyl map*, which is the functional map from function F to the corresponding operator.

$$\hat{\mathcal{W}}[F] \equiv \int d^D x\ F(x)\hat{\Delta}(x) \tag{2.12}$$

This can be equivalently written as follows.

$$\hat{\mathcal{W}}[F] = \frac{1}{(2\pi)^D}\int d^D k\ \tilde{F}(k) \exp\left(ik_\mu \hat{x}^\mu\right), \tag{2.13}$$

where $\tilde{F}$ denotes the Fourier function of F,

$$\tilde{F}(k) = \int dx^D F(x) \exp\left(-ik_\mu x^\mu\right). \tag{2.14}$$

We can regard $\hat{\mathcal{W}}[F]$ as the operator of F in the coordinate representation.

To define a meaningful theory, especially a local field theory in Lagrangian formalism, well-defined procedures of differentiation and integration are needed. In usual (commutative) physics, fields are considered as the (often infinitely) differentiable sections on a differentiable manifold. We are free to perform mathematical infinitesimal operations on the functions or the space. However that is not the case with noncommutative spacetime. Since noncommutativity (2.8) works as the uncertainty relation between coordinates, the word *point* is meaningless in noncommutative space.

The differential calculus in noncommutative space is achieved as to maintain their algebraic structure of that in commutative space. In particular we define the "derivative" operator $\hat{\partial}$, which is a anti-Hermitian operator and works on operators such as,

$$\left[\hat{\partial}_\mu, \hat{x}^\nu\right] = \delta_\mu^{\ \nu}, \tag{2.15}$$

$$\left[\hat{\partial}_\mu, \hat{\partial}_\nu\right] = 0. \tag{2.16}$$

With short calculation, we find that this $\hat{\partial}$ act on $\hat{\Delta}(x)$ such that

$$\left[\hat{\partial}_\mu, \hat{\Delta}(x)\right] = -\partial_\mu \hat{\Delta}(x). \tag{2.17}$$

It shows that $\hat{\partial}$ works as the exact derivative operator for all function F.

$$\left[\hat{\partial}_\mu, \hat{\mathcal{W}}[F]\right] = \int d^D \partial_\mu F(x) \hat{\Delta}(x) = \hat{\mathcal{W}}[\partial_\mu F] \tag{2.18}$$

Here we use partial integration. The surface integral is dropped with the condition (2.9).

The trace of the operator space may be defined with the normalization Tr $\hat{\Delta}(x) = 1$. This trace allows us the inverse maps,

$$\int d^D F(x) = \mathrm{Tr} \hat{\mathcal{W}}[F], \tag{2.19}$$

and

$$F(x) = \mathrm{Tr}\left(\hat{\mathcal{W}}[F]\hat{\Delta}(x)\right). \tag{2.20}$$

The product of functions F and G in noncommutative space is given by the corresponding operators product $\hat{\mathcal{W}}[F]\hat{\mathcal{W}}[G]$. We can derive its inverse,

$$\mathrm{Tr}\left(\hat{\mathcal{W}}[F]\hat{\mathcal{W}}[G]\hat{\Delta}(x)\right) = \frac{1}{\pi^D \mid \det \Theta \mid} \iint d^D y d^D z F(y) G(z) e^{-2i(\Theta^{-1})_{\mu\nu}(x-y)^\mu (x-z)^\nu}. \tag{2.21}$$

Here we used

$$\hat{\Delta}(x)\hat{\Delta}(y) = \frac{1}{\pi^D \ |\det\Theta|}\int d^D z\hat{\Delta}(z)e^{-2i(\Theta^{-1})_{\mu\nu}(x-z)^\mu(y-z)^\nu} \tag{2.22}$$

and

$$\mathrm{Tr}\left(\hat{\Delta}(x)\hat{\Delta}(y)\right) = \delta^D(x-y). \tag{2.23}$$

Note that here we assumed that $\Theta^{\mu\nu}$ is an invertible matrix. Otherwise we have to redefine x^μ to eliminate the commutative subset and down the size of the matrix, if needed, to keep $\Theta^{\mu\nu}$ invertible.

From these relations, we see

$$\hat{\mathcal{W}}[F]\hat{\mathcal{W}}[G] = \hat{\mathcal{W}}[F\star G], \tag{2.24}$$

here the star product $\star$ is the *Moyal product* which defined as

$$\star \equiv \exp\left(\frac{i}{2}\overleftarrow{\partial_\mu}\Theta^{\mu\nu}\overrightarrow{\partial_\nu}\right). \tag{2.25}$$

The Moyal product gives an associative but noncommutative product of functions as follows.

$$\begin{aligned} F\star G &= F(x)\exp\left(\frac{i}{2}\overleftarrow{\partial_\mu}\Theta^{\mu\nu}\overrightarrow{\partial_\nu}\right)G(x) \\ &= F(x)G(x) + \sum_{n=1}^{\infty}\frac{i}{2}^n\frac{1}{n!}\Theta^{\mu_1\nu_1}\cdots\Theta^{\mu_n\nu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}F(x)\ \partial_{\nu_1}\cdots\partial_{\nu_n}G(x) \end{aligned} \tag{2.26}$$

The higher order product is also derived such that

$$\mathrm{Tr}\left(\hat{\mathcal{W}}[F_1]\cdots\hat{\mathcal{W}}[F_n]\right) = \int d^D x F_1(x)\star\cdots\star F_n(x). \tag{2.27}$$

Thus, we reach the following conclusion. The theory in the noncommutative spacetime is constructed in operator formalism with the Weyl mapping, written by the operator function of coordinate representation, with the help of well-defined trace Tr and a derivative operator $\hat{\partial}$. The integral and derivative in commutative spacetime are replaced with the trace and derivative operator on the Hilbert space, respectively.

Whereas that is equivalent to the theory written by the usual function on commutative space, but only with the multiplication rule is modified, namely with the Moyal product. In general, to construct a theory on the noncommutative spacetime comes down to the modification of product, which we call a *star product*. To construct a noncommutative theory, in a commutative field theory descriptions, is equivalent to finding the suitable star product. For the Weyl quantization, the Moyal product is the right star product.

The commutator with the Moyal product provides desirable noncommutative coordinate relation,

$$\begin{aligned} [x^\mu, x^\nu]_\star &\equiv x^\mu \star x^\nu - x^\nu \star x^\mu \\ &= i\Theta^{\mu\nu}. \end{aligned} \tag{2.28}$$

If we require only the relation (2.28), the relevant term in the Moyal product (2.25) is at the order $\mathcal{O}(\Theta^1)$. Other higher terms do not change the relation (2.28), therefore we have the freedom of determining their coefficients. It means that there are many noncommutative theories on the space with same noncommutativity. It is a little bit similar situation in the case of the ordering problem in quantum mechanics. But if we impose associativity of the product at all order of Θ, then the higher order terms are determined uniquely such as the Moyal product. This is another reason that the Moyal product is a suitable star product.

From Eq.(2.27) and the cyclic property of trace, the integration of the product of functions with the Moyal product is invariant under a cyclic rotation too.

$$\int d^D x F_1(x) \star \cdots \star F_{n-1}(x) \star F_n(x) = \int d^D x F_n(x) \star F_1(x) \star \cdots \star F_{n-1}(x) \tag{2.29}$$

From this, or directly by using partial integration, we find the integral of a biproduct is not deformed with the star product.

$$\int d^D x F(x) \star G(x) = \int d^D x F(x) G(x). \tag{2.30}$$

Now we can formulate the field theory on noncommutative spacetime. For example, the action of the real scalar ϕ^4 theory on the noncommutative space is given as follows.

$$
\begin{aligned}
S[\phi] &= \int d^D x \left[\frac{1}{2}(\partial^\mu \phi) \star (\partial_\mu \phi) + \frac{m^2}{2}\phi \star \phi + \frac{g^2}{4!}\phi \star \phi \star \phi \star \phi\right] \\
&= \int d^D x \left[\frac{1}{2}(\partial^\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g^2}{4!}\phi \star \phi \star \phi \star \phi\right]
\end{aligned} \tag{2.31}
$$

In such deformation of noncommutative theory, generally nontrivial contributions come from the terms which have the product of third or more higher order.

Because the Moyal product is defined by the infinite series of derivative operators, derivatives of fields at all order appear in the action, thus the theory becomes nonlocal. This is a characteristic feature of the theory on noncommutative space.

Many interesting features of the field theories on noncommutative space, for example UV/IR mixing, gauge theory and Seiberg-Witten map, etc. have been known, but these are beyond the scope of this thesis[3].

2.3 $\mathcal{N} = 1/2$ Supersymmetric Theory in Nonanticommutative Superspace

Supersymmetric theory is most naturally formulated in superspace. $\mathcal{N} = 1$ superspace is defined the set of coordinate $(x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}})$, which consists of usual spacetime coordinate x^μ and additional fermionic (anticommuting numbers) coordinates with spinorial indices θ_α and $\bar{\theta}^{\dot{\alpha}}$.

Recently the realization of noncommutativity in superspace as the effective description in low energy region is found from superstring theory. Nonanticommutativity of fermionic coordinate θ^α in $\mathcal{N} = 1$ four-dimensional superspace arises in the existence of constant graviphoton background [20, 19, 21].

Such nonanticommutativity is formulated in supersymmetric QFT on four-dimensional superspace. We review it following Seiberg[19].

[3] For further details of noncommutative field theory itself, good reviews are available[10, 9].

We consider that the chiral part of superspace coordinate θ^α becomes nonanticommutative, which obeys a Clifford algebra rather than an anticommutative relation,

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}, \tag{2.32}$$

while anticommutators for anti-chiral partner $\bar{\theta}^{\dot{\alpha}}$ are not modified,

$$\{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = 0 \quad \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0. \tag{2.33}$$

Here $C^{\alpha\beta}$ is a constant parameter with symmetric indices, and correspond naively to the VEV of constant graviphoton background field.

From these relations $\bar{\theta}^{\dot{\alpha}}$ can not be taken as the complex conjugate of θ^α,

$$(\theta^\alpha)^\dagger \neq \bar{\theta}^{\dot{\alpha}}. \tag{2.34}$$

Therefore the theory can be formulated only in Euclidean spacetime or Atiyah-Ward spacetime.

Nonanticommutative theory in superspace can also be formulated by the modification of the multiplication rule. The star product is given as the exponential function of derivative with respect to fermionic coordinates like Moyal product,

$$\star = \exp\left(\frac{-C^{\alpha\beta}}{2}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial\theta^\beta}\right), \tag{2.35}$$

where a right and left derivative conventions are

$$\frac{\overrightarrow{\partial}}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha^{\ \beta}, \tag{2.36}$$

$$\theta^\beta\frac{\overleftarrow{\partial}}{\partial\theta^\alpha} = -\delta^\beta_{\ \alpha}. \tag{2.37}$$

This star product gives the nonanticommutative relation (2.32).

Because of nilpotency of θ derivatives, the expansion series of the exponential function in the star product Eq. (2.35) terminates at finite order.

$$\begin{aligned} F(\theta)\star G(\theta) &= F(\theta)\exp\left(-\frac{C^{\alpha\beta}}{2}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial\theta^\beta}\right)G(\theta) \\ &= F(\theta)\left(1-\frac{C^{\alpha\beta}}{2}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial\theta^\beta} - \det C\,\frac{\overleftarrow{\partial}}{\partial\theta\theta}\frac{\overrightarrow{\partial}}{\partial\theta\theta}\right)G(\theta), \end{aligned} \tag{2.38}$$

where we defined

$$\frac{\partial}{\partial\theta\theta} \equiv \frac{1}{4}\varepsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}. \tag{2.39}$$

The number of the terms in the star product is finite and all derivatives are with respect to θ^α. Obviously no spacetime derivative is included, therefore this noncommutativity holds locality of the theory. That is a different point from spacetime noncommutativity.

The commutation relations of bosonic coordinate have not yet been fixed. We take chiral coordinate $y^\mu \equiv x^\mu - i\theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$ as an independent variable instead of x^μ, and imply

$$[y^\mu, y^\nu] = [y^\mu, \theta^\alpha] = [y^\mu, \bar{\theta}^{\dot{\alpha}}] = 0, \tag{2.40}$$

instead of $[x^\mu, x^\nu] = 0$, $[x^\mu, \theta^\alpha] = 0$ and so on. This situation correctly agrees with the nonanticommutativity which is realized from string theory. Moreover, to use the chiral coordinate as an independent variable helps us to define a chiral and antichiral superfield. The commutators of x are modified in this case,

$$\begin{aligned}\left[x^\mu, \theta^\beta\right] &= -iC^{\alpha\beta}\sigma^\mu{}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}, \\ [x^\mu, x^\nu] &= C^{\alpha\beta}\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\sigma^\nu{}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}.\end{aligned} \tag{2.41}$$

In chiral coordinate base, the representations of supercharges Q and super covariant derivatives D are given as the differential operators on superspace as follows.

$$\begin{aligned}Q_\alpha &= i\frac{\partial}{\partial\theta^\alpha} \\ \bar{Q}_{\dot{\alpha}} &= -i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + 2\theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu} \\ D_\alpha &= i\frac{\partial}{\partial\theta^\alpha} + 2\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^\mu} \\ \bar{D}_{\dot{\alpha}} &= -i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\end{aligned} \tag{2.42}$$

Notice that all partial differential operations are done with respect to independent variables $(y^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}})$, not $(x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}})$.

Under the conditions of Eq.(2.32), (2.33) and Eq.(2.40), commutation relations between Q and D are calculated.

$$
\begin{aligned}
\{Q_\alpha, Q_\beta\} &= 0 \\
\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2\sigma^\mu{}_{\alpha\dot{\alpha}} \frac{\partial}{\partial y^\mu} \\
\{D_\alpha, D_\beta\} &= \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \\
\{D_\alpha, \bar{D}_{\dot{\alpha}}\} &= -2\sigma^\mu{}_{\alpha\dot{\alpha}} \frac{\partial}{\partial y^\mu} \\
\{D_\alpha, Q_\beta\} &= \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\alpha}}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\alpha}}\} = 0
\end{aligned}
\tag{2.43}
$$

Above relations are the same as the ones in commutative superspace. But only $\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\alpha}}\}$ is modified,

$$
\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 4C^{\alpha\beta}\sigma^\mu{}_{\alpha\dot{\alpha}}\sigma^\nu{}_{\beta\dot{\beta}} \frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}. \tag{2.44}
$$

This commutator breaks the supersymmetry algebra, thus $\bar{Q}_{\dot{\alpha}}$ is no longer the symmetry in nonanticommutative superspace while Q_α is still the symmetry. For this reason, nonanticommutative theory of such deformation is called $\mathcal{N} = 1/2$ supersymmetric theory. In the commutative limit $C^{\alpha\beta} \to 0$, all commutation relations are reduced to the usual ones, *i.e.*, supersymmetry is recovered.

Note that we can rewrite the star product (2.35) such that

$$
\star = \exp\left(\frac{C^{\alpha\beta}}{2}\overleftarrow{Q}_\alpha \overrightarrow{Q}_\alpha\right). \tag{2.45}
$$

From this form, it is clear that the star product maps the product of chiral superfields to a chiral superfield, and maps the product of antichiral superfields to an antichiral superfield.

3 Fermionic-Bosonic Mixed Noncommutative Superspace

We can consider another noncommutative relation between superspace coordinates, namely between a spacetime coordinate and fermionic coordinate, as follows.

$$[x^\mu, \theta^\alpha] = i\lambda^{\mu\alpha}, \tag{3.1}$$

where $\lambda^{\mu\alpha}$ is a constant fermionic number, *i.e.* Grassmann number constant, which has both a spacetime index and spinor index.

The possibility of noncommutative relation (3.1) is suggested in several literatures. D. Klemm *et al.* [22] argued that the noncommutativity (3.1) is allowed from the algebraic point of view. de Boer *et al.*[23] studied string theory in nontrivial background fields of 10-dimensional $\mathcal{N} = 2$ supergravity and found that the noncommutativity (3.1) corresponds to the VEV of gravitino field. Ferrara *et al.*[24] formulated the theory on four dimensional noncommutative superspace which realize the relation (3.1) with the Moyal product, and gave a deformed Lagrangian. But almost no further serious investigations have been done.

The work[25] is the first attempt of a rigorous study of such noncommutativity in $\mathcal{N} = 1$ four-dimensional noncommutative superspace. The theory in the noncommutativity (3.1) has an intermediate nature between the spacetime noncommutativity and the fermionic coordinate noncommutativity like $\mathcal{N} = 1/2$ SUSY theory. In this section, we will introduce the noncommutative theory on this Fermionic-Bosonic Mixed Noncommutative Superspace in some detail.

3.1 Formulation with a Moyal Product

D. Klemm *et al.*[22] investigated the algebraic consistency of noncommutativity in four-dimensional superspace. They started with the general setup in $\mathcal{N} = 1$ four-dimensional superspace, where all noncommutative relation between superspace coordinates are arbitrary, and their noncommutativity parameters can depend on all superspace coordinates. Under the condition that noncommutativity parameters are invariant under the supertranslation,

$$\begin{aligned}
\theta'^{\alpha} &= \theta^{\alpha} + \epsilon^{\alpha}, \\
\bar{\theta}^{\dot{\alpha}'} &= \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}}, \\
x'^{\mu} &= x^{\mu} + a^{\mu} + i(\epsilon^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}}\sigma^{\mu}_{\alpha\dot{\alpha}}\theta^{\alpha}), \qquad (3.2)
\end{aligned}$$

where ϵ^{α}, $\bar{\epsilon}^{\dot{\alpha}}$ and a^{μ} are infinitesimal translation parameters, and the star product is associative, they found that the noncommutative relation (3.1) is allowed in four-dimensional Minkowski $\mathcal{N} = 1$ superspace. They pointed out that it is also allowed in $\mathcal{N} = 2$ Euclidean superspace. That is in contrast to the case of the noncommutativity $\{\theta^{\alpha}, \theta^{\beta}\} \neq 0$, which is allowed only in Euclidean spacetime.

Based on this fact, we try to formulate the noncommutative theory with an appropriate Moyal product which has associative property, in both Mankowski and Euclidean spacetime. We take chiral coordinate as an independent variable as in the case of the formulation for nonanticommutative superspace by Seiberg[19]. However, in this case we have no implications from string theory, therefore there is no unique way in writing the theory in the chiral coordinate base. It depends on how will we define and construct the theory. Here we demand that it simply maintain well-defined chiral superfields.

We found the appropriate star product,

$$\star = \exp\left[\frac{i}{2}\lambda^{\mu\alpha}\left(\frac{\overleftarrow{\partial}}{\partial y^{\mu}}\frac{\overrightarrow{\partial}}{\partial \theta^{\alpha}} - \frac{\overleftarrow{\partial}}{\partial \theta^{\alpha}}\frac{\overrightarrow{\partial}}{\partial y^{\mu}}\right)\right]. \qquad (3.3)$$

This star product gives the desired noncommutative relation.

$$[y^{\mu}, \theta^{\alpha}]_{\star} = [x^{\mu}, \theta^{\alpha}]_{\star} = i\lambda^{\mu\alpha} \qquad (3.4)$$

We can keep the following relations without modification,

$$\begin{aligned}
[y^{\mu}, y^{\nu}] &= 0, \\
\{\theta^{\alpha}, \theta^{\beta}\} &= \{\theta^{\alpha}, \bar{\theta}^{\dot{\beta}}\} = \{\bar{\theta}^{\dot{\beta}}, \bar{\theta}^{\dot{\beta}}\} = 0. \qquad (3.5)
\end{aligned}$$

However there are two ways to demand other noncommutative relations. In Minkowski, we have to keep hermiticity of the theory. Thus the Hermitian conjugate of Eq.(3.4) implies

$$[\bar{y}^\mu, \bar{\theta}^{\dot{\alpha}}]_\star = i\bar{\lambda}^{\mu\dot{\alpha}}, \tag{3.6}$$

where $\bar{y}^\mu = (y^\mu)^\dagger = x^\mu + i\theta^\alpha \sigma^\mu{}_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}$. This relation cannot be realized by the only one star product (3.4). Consider Hermitian conjugate of the product of two chiral superfields, $(f(y,\theta) \star g(y,\theta))^\dagger = \bar{g}(\bar{y},\bar{\theta}) \star^\dagger \bar{f}(\bar{y},\bar{\theta})$. Such term appears inevitably in Lagrangian if we construct the Hermitian theory, therefore the Hermitian conjugate of the star product is needed, since $\star^\dagger \neq \star$. Formally it is written,

$$\begin{aligned} \bar{\star} &\equiv \star^\dagger \\ &= \exp\left[\frac{i}{2}\bar{\lambda}^{\mu\dot{\alpha}}\left(\frac{\overleftarrow{\partial}}{\partial\bar{y}^\mu}\frac{\overrightarrow{\partial}}{\partial\bar{\theta}^{\dot{\alpha}}} - \frac{\overleftarrow{\partial}}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\overrightarrow{\partial}}{\partial\bar{y}^\mu}\right)\right]. \end{aligned} \tag{3.7}$$

We assume that $\bar{\star}$ is used for products of anti-chiral superfield, instead of $\star$.

On the other hand, in Euclidean spacetime $\bar{\theta}^{\dot{\alpha}}$ is not necessary to be the hermitian conjugate of θ^α. Thus we can formulate the theory with one star product.

Thanks to nilpotency of the derivative with respect to fermionic coordinate $\frac{\partial}{\partial\theta^\alpha}$, Taylor expansion series in the star product (3.3) terminates at finite order,

$$\begin{aligned} \star &= \exp\left[\frac{i}{2}\lambda^{\mu\alpha}\left(\frac{\overleftarrow{\partial}}{\partial y^\mu}\frac{\overrightarrow{\partial}}{\partial\theta^\alpha} - \frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial y^\mu}\right)\right] \\ &= 1 + \frac{i}{2}\lambda^{\mu\alpha}\left(\frac{\overleftarrow{\partial}}{\partial y^\mu}\frac{\overrightarrow{\partial}}{\partial\theta^\alpha} - \frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial y^\mu}\right) \\ &\quad + \frac{1}{8}\lambda^{\mu\alpha}\lambda^{\nu\beta}\left(\frac{\overleftarrow{\partial}}{\partial y^{\mu\nu}}\frac{\overrightarrow{\partial}}{\partial\theta^{\alpha\beta}} + 2\frac{\overleftarrow{\partial}}{\partial y^\mu\partial\theta^\beta}\frac{\overrightarrow{\partial}}{\partial\theta^\alpha\partial y^\nu} + \frac{\overleftarrow{\partial}}{\partial\theta^{\alpha\beta}}\frac{\overrightarrow{\partial}}{\partial y^{\nu\mu}}\right) \\ &\quad + \frac{i}{16}\lambda^{\mu\alpha}\lambda^{\nu\beta}\lambda^{\rho\gamma}\left(\frac{\overleftarrow{\partial}}{\partial y^\rho\partial\theta^{\alpha\beta}}\frac{\overrightarrow{\partial}}{\partial\theta^\gamma\partial y^{\nu\mu}} - \frac{\overleftarrow{\partial}}{\partial y^{\mu\rho}\partial\theta^\beta}\frac{\overrightarrow{\partial}}{\partial\theta^{\gamma\alpha}\partial y^\nu}\right) \\ &\quad + \frac{1}{64}\lambda^{\mu\alpha}\lambda^{\nu\beta}\lambda^{\rho\gamma}\lambda^{\sigma\delta}\frac{\overleftarrow{\partial}}{\partial y^{\mu\rho}\partial\theta^{\beta\delta}}\frac{\overrightarrow{\partial}}{\partial\theta^{\gamma\alpha}\partial y^{\sigma\nu}}. \end{aligned} \tag{3.8}$$

Here we use the abbreviations $\frac{\overrightarrow{\partial}^2}{\partial y^\mu\partial y^\nu} \equiv \frac{\overrightarrow{\partial}}{\partial y^{\mu\nu}}$ and $\frac{\overrightarrow{\partial^2}}{\partial\theta^\alpha\partial\theta^\beta} \equiv \frac{\overrightarrow{\partial}}{\partial\theta^{\alpha\beta}}$.

3.2 Noncommutative Deformation of Wess-Zumino model

In the following we investigate the supersymmetric quantum field theory with a simple example, the Wess-Zumino model.

A chiral super field Φ is defined with (anti)super covariant derivative $\bar{D}_{\dot{\alpha}}$, such that $\bar{D}_{\dot{\alpha}}\Phi = 0$. Φ can be written as

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta^{\alpha}\psi_{\alpha}(y) + \theta^2 F(y). \tag{3.9}$$

In this noncommutative deformation, a chiral superfield is well-defined. The star product (3.3) can be rewritten such as

$$\star = \exp\left[\frac{i}{2}\lambda^{\mu\alpha}\left(\frac{\overleftarrow{\partial}}{\partial y^{\mu}}\overrightarrow{Q_{\alpha}} - \overleftarrow{Q_{\alpha}}\frac{\overrightarrow{\partial}}{\partial y^{\mu}}\right)\right]. \tag{3.10}$$

This star product clearly commutes with super-covariant derivatives D_{α} and $\bar{D}_{\dot{\alpha}}$, therefore a product of chiral superfields $\Phi \star \Phi'$ is a chiral superfields again, and a product of antichiral superfields is an antichiral superfield too.

The Lagrangian of the simplest Wess-Zumino model in commutative superspace is given by

$$\mathcal{L}_0 = \int d^4\theta\ \overline{\Phi}\Phi + \int d^2\theta\ \left(\frac{m}{2}\Phi\Phi + \frac{g}{3}\Phi\Phi\Phi\right) + \int d^2\bar{\theta}\ \left(\frac{\bar{m}}{2}\overline{\Phi}\,\overline{\Phi} + \frac{\bar{g}}{3}\overline{\Phi}\,\overline{\Phi}\,\overline{\Phi}\right). \tag{3.11}$$

Here m and g are a mass parameter and a coupling constant, respectively. We derive the noncommutative deformed Lagrangian by replacing all products with the star products in Eq.(3.11).

3.2.1 Deformed Lagrangian in Euclidean space

As mentioned above, in Euclidean space we can use one star product for both chiral and antichiral part.

$$\begin{aligned}\mathcal{L} \ &= \ \int d^4\theta\ \overline{\Phi}\star\Phi + \int d^2\theta\ \left(\frac{m}{2}\Phi\star\Phi + \frac{g}{3}\Phi\star\Phi\star\Phi\right) \\ &\quad + \int d^2\bar{\theta}\ \left(\frac{\bar{m}}{2}\overline{\Phi}\star\overline{\Phi} + \frac{\bar{g}}{3}\overline{\Phi}\star\overline{\Phi}\star\overline{\Phi}\right).\end{aligned} \tag{3.12}$$

An explicit expansion of the star product shows the deformed action in the component fields,

$$
\begin{aligned}
S &= \int d^4x \mathcal{L}_0 + \frac{i}{\sqrt{2}}\lambda^{\mu\alpha}\psi_\alpha(\partial_\mu \Box \bar{A}) + \frac{1}{\sqrt{2}}\lambda^{\mu\alpha}F(\sigma^\nu)_{\alpha\dot{\alpha}}(\partial_\mu\partial_\nu\bar{\psi}^{\dot{\alpha}}) - \frac{1}{2}\lambda^{\mu\nu}F(\partial_\mu\partial_\nu\Box\bar{A}) \\
&\quad + \frac{g}{2}\lambda^{\mu\alpha}\lambda^{\nu\beta}F(\partial_\mu\psi_\beta)(\partial_\nu\psi_\alpha) + g\lambda^{\mu\nu}F(\partial_\mu F)(\partial_\nu A) + \frac{g}{3}\lambda^{\mu\nu\rho\sigma}F(\partial_\mu\partial_\nu F)(\partial_\rho\partial_\sigma F) \\
&\quad + \frac{\bar{g}}{3}\lambda^{\mu\nu}\left[-(\partial_\mu\bar{A})(\partial_\nu\bar{A})(\Box\bar{A}) + (\partial_\rho\bar{A})(\partial^\rho\bar{A})(\partial_\mu\partial_\nu\bar{A})\right]. \qquad (3.13)
\end{aligned}
$$

Here we have integrated out Grassmann coordinates and dropped total derivative terms, and $\Box$ denotes d'Alembertian operator $\partial^\mu\partial_\mu$. The abbreviations for noncommutative parameter $\lambda^{\mu\alpha}$ are given in the Appendix. $\mathcal{L}_0$ is the undeformed part of the Lagrangian which is the same as the original Wess-Zumino model. In addition to the original Wess-Zumino Lagrangian, there appear higher derivative terms which are of finite order, rather than infinite order as is the case for spacetime noncommutativity.

This action is not Hermitian because the terms added are obviously not Hermitian. We will not go further in Euclidean case, because non-Hermitian Lagrangian is difficult to compare with a realistic theory. Since the formulation of the noncommutative deformed Wess-Zumino model is allowed in Minkowski space, we focus on the theory in Minkowski space and investigate it hereafter.

3.2.2 Deformed Lagrangian in Minkowski space

In Minkowski space, we have to deform the Lagrangian with $\star$ for the chiral part, and with $\bar{\star}$ for antichiral part,

$$
\begin{aligned}
\mathcal{L} &= \int d^4\theta\ \overline{\Phi} \star \Phi + \int d^2\theta\ \left(\frac{m}{2}\Phi \star \Phi + \frac{g}{3}\Phi \star \Phi \star \Phi\right) \\
&\quad + \int d^2\bar{\theta}\ \left(\frac{\bar{m}}{2}\overline{\Phi}\ \bar{\star}\ \overline{\Phi} + \frac{\bar{g}}{3}\overline{\Phi}\ \bar{\star}\ \overline{\Phi}\ \bar{\star}\overline{\Phi}\right). \qquad (3.14)
\end{aligned}
$$

The Kähler term is slightly complicated, because it is a mixed term of chiral and antichiral superfields, thus it produces ambiguity of the choice of the star product. This ambiguity is not discussed here, and we only use the fact that a biproduct in the integration is not modified with the star product,

$$\int d^4\theta\ \overline{\Phi}\star\Phi = \int d^4\theta\ \overline{\Phi}\bar{\star}\Phi = \int d^4\theta\ \overline{\Phi}\Phi. \tag{3.15}$$

Thus Lagrangian is clearly Hermitian.

Explicit calculation gives

$$\begin{aligned} S &= \int d^4x\ \mathcal{L}_0 + \frac{g}{3}\int d^4x\ \Big[\lambda^{\mu\alpha}\lambda^{\nu\beta}\left(\psi_\alpha(\partial_\mu\psi_\beta)(\partial_\nu F) + \frac{1}{2}F(\partial_\mu\psi_\beta)(\partial_\nu\psi_\alpha)\right) \\ &\quad +\lambda^{\mu\nu}\left(F\psi^\alpha(\partial_\mu\partial_\nu\psi_\alpha) - AF(\partial_\mu\partial_\nu F) - A(\partial_\mu F)(\partial_\nu F) - F^2(\partial_\mu\partial_\nu A)\right) \\ &\quad +\lambda^{\mu\nu\rho\sigma}F(\partial_\mu\partial_\nu F)(\partial_\rho\partial_\sigma F)\Big] + [(\text{h.c.})]. \end{aligned} \tag{3.16}$$

The undeformed part of the Wess-Zumino Lagrangian written in component fields is

$$\begin{aligned} \mathcal{L}_0 &= \bar{A}\Box A + i\partial_\mu\bar{\psi}_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}\psi_\beta + \bar{F}F \\ &\quad + \left[m(AF - \frac{1}{2}\psi\psi) + g(A^2F - \psi\psi A) + (\text{h.c.})\right]. \end{aligned} \tag{3.17}$$

The deformed action (3.16) is no longer supersymmetric, but has a remaining symmetry. The undeformed part and λ-dependent part (not depend on $\bar\lambda$) is invariant under the Q-supersymmetry transformation,

$$\begin{aligned} \delta_\xi A &= \sqrt{2}\xi\psi, \\ \delta_\xi\psi &= \sqrt{2}\xi F, \\ \delta_\xi F &= 0. \end{aligned} \tag{3.18}$$

The invariance of the λ-dependent part can be proved by an explicit calculation.

$$\begin{aligned} \frac{3}{g}\delta\mathcal{L}_\lambda &= \sqrt{2}\lambda^{\mu\alpha}\lambda^{\nu\beta}\left(\xi_\alpha F\partial_\mu\psi_\beta\partial_\nu F + \psi_\alpha\partial_\mu\xi_\beta F\partial_\nu F + \frac{1}{2}F\partial_\mu\xi_\beta F\partial_\nu\psi_\alpha + \frac{1}{2}F\partial_\mu\psi_\beta\partial_\nu\xi_\alpha F\right) \\ &\quad -\sqrt{2}\lambda^{\mu\nu}\left(-F\xi^\alpha F\partial_\mu\partial_\nu\psi_\alpha - F\psi^\alpha\partial_\mu\partial_\nu\xi_\alpha F + \xi^\alpha\psi_\alpha F\partial_\mu\partial_\nu F\right. \\ &\qquad\qquad \left. +\xi^\alpha\psi_\alpha\partial_\mu F\partial_\nu F + F^2\partial_\mu\partial_\nu\xi^\alpha\psi_\alpha\right) \\ &= \sqrt{2}\lambda^{\mu\alpha}\lambda^{\nu\beta}\psi_\alpha\xi_\beta\partial_\mu F\partial_\nu F - \sqrt{2}\lambda^{\mu\nu}\xi^\alpha\psi_\alpha\partial_\mu F\partial_\nu F \\ &= 0, \end{aligned} \tag{3.19}$$

where we have used $\lambda^{\mu\alpha}\partial_\mu F\lambda^{\nu\beta}\partial_\nu F = -\varepsilon^{\alpha\beta}\lambda^{\mu\nu}\partial_\mu F\partial_\nu F$.

However the λ-dependent part is not invariant under the $\bar{Q}$-supersymmetry transformation. On the contrary, $\bar{\lambda}$-dependent part is $\bar{Q}$-supersymmetric but not Q-supersymmetric. Of course the undeformed part is invariant under both the Q-supersymmetry and $\bar{Q}$-supersymmetry transformation. Since the whole action cannot be invariant under the Q-supersymmetry and $\bar{Q}$-supersymmetry transformation simultaneously, supersymmetry is completely broken, namely $\mathcal{N} = 0$.

But there is a remaining Boson-Fermion symmetry,

$$\begin{aligned} \delta_\xi A &= \sqrt{2}\xi\psi, & \delta_{\bar{\xi}}\bar{A} &= \sqrt{2}\bar{\xi}\bar{\psi}, \\ \delta_\xi \psi &= \sqrt{2}\xi F, & \delta_{\bar{\xi}}\bar{\psi} &= \sqrt{2}\bar{\xi}\bar{F}, \\ \delta_\xi F &= 0, & \delta_{\bar{\xi}}\bar{F} &= 0. \end{aligned} \tag{3.20}$$

Next we consider the auxiliary fields. The generalized equation of motion for an auxiliary field $\bar{F}$ is

$$\begin{aligned} F &= -(\bar{m}\bar{A} + \bar{g}\bar{A}^2) \\ &\quad + \bar{g}\bar{\lambda}^{\mu\nu}\bar{F}\partial_\mu\partial_\nu\bar{A} - \frac{1}{2}\bar{g}\bar{\lambda}^{\mu\dot{\alpha}}\bar{\lambda}^{\nu\dot{\beta}}(\partial_\mu\bar{\psi}_{\dot{\beta}}\partial_\nu\bar{\psi}_{\dot{\alpha}}) \\ &\quad - \frac{1}{3}\bar{g}\bar{\lambda}^{\mu\nu\rho\sigma}\left\{(\partial_\mu\partial_\nu\bar{F})(\partial_\rho\partial_\sigma\bar{F}) + 2\partial_{\mu\nu}(\bar{F}\partial_{\rho\sigma}\bar{F})\right\}, \end{aligned} \tag{3.21}$$

and a similar equation for F, which is the Hermite conjugate of Eq.(3.21). In usual supersymmetric theory, the auxiliary fields have no degrees of freedom, because we can eliminate them by the equation of motions. However, in this case, it is not clear if that is the case. The non-commutative deformed equation of motion (3.21) is more complicated, and contains derivative terms of $\bar{F}$. But Eq.(3.21) can be solved easily due to nilpotency of Grassmann constant λ, with the following trick. We eliminate $\bar{F}$ in Eq.(3.21) using the equation of other Hermitian conjugate half of Eq.(3.21),

$$
\begin{aligned}
\bar{F} &= -(mA + gA^2) \\
&\quad + g\lambda^{\mu\nu} F \partial_\mu \partial_\nu A - \frac{1}{2} g \lambda^{\mu\alpha} \lambda^{\nu\beta} (\partial_\mu \psi_\beta \partial_\nu \psi_\alpha) \\
&\quad - \frac{1}{3} g \lambda^{\mu\nu\rho\sigma} \left\{ (\partial_\mu \partial_\nu F)(\partial_\rho \partial_\sigma F) + 2\partial_{\mu\nu}(F \partial_{\rho\sigma} F) \right\},
\end{aligned}
\tag{3.22}
$$

and get the differential equation in terms of F only.

$$
\begin{aligned}
F &= -(\bar{m}\bar{A} + \bar{g}\bar{A}^2) \\
&\quad - \frac{1}{2} \bar{g} \bar{\lambda}^{\mu\dot{\alpha}} \bar{\lambda}^{\nu\dot{\beta}} (\partial_\mu \bar{\psi}_{\dot{\beta}} \partial_\nu \bar{\psi}_{\dot{\alpha}}) \\
&\quad + \bar{g} \bar{\lambda}^{\mu\nu} (\bar{m}\bar{A} + \bar{g}\bar{A}^2) \partial_\mu \partial_\nu \bar{A} + \cdots \\
&\quad + g\bar{g} \lambda^{\mu\nu} \bar{\lambda}^{\rho\sigma} F (\partial_\mu \partial_\nu A)(\partial_\rho \partial_\sigma \bar{A}) + \cdots
\end{aligned}
\tag{3.23}
$$

We can replace F in RHS with the whole expression of F itself,

$$
\begin{aligned}
& g\bar{g} \lambda^{\mu\nu} \bar{\lambda}^{\rho\sigma} F (\partial_\mu \partial_\nu A)(\partial_\rho \partial_\sigma \bar{A}) \longrightarrow \\
& \quad g\bar{g} \lambda^{\mu\nu} \bar{\lambda}^{\rho\sigma} \Big(-(\bar{m}\bar{A} + \bar{g}\bar{A}^2) - \frac{1}{2} \bar{g} \bar{\lambda}^{\mu\dot{\alpha}} \bar{\lambda}^{\nu\dot{\beta}} (\partial_\mu \bar{\psi}_{\dot{\beta}} \partial_\nu \bar{\psi}_{\dot{\alpha}}) \\
& \quad + \bar{g} \bar{\lambda}^{\mu\nu} (\bar{m}\bar{A} + \bar{g}\bar{A}^2) \partial_\mu \partial_\nu \bar{A} + \cdots \Big) (\partial_\mu \partial_\nu A)(\partial_\rho \partial_\sigma \bar{A})
\end{aligned}
\tag{3.24}
$$

This replacement procedure should be done iteratively. Usually this operation only results in giving a higher order differential equation in terms of F. However in this case, higher derivative terms of F involve Grassmann number parameters λ and/or $\bar{\lambda}$. These terms will vanish at most at the order $\mathcal{O}(\lambda^8)$ or $\mathcal{O}(\bar{\lambda}^8)$, therefore the procedure will terminate in a finite number of times. In the end, all F in LHS are eliminated, and F is represented in terms of dynamical component fields only, *i.e.*, F is solved. From this cause, we conclude that F always remains an auxiliary field. In other word, introducing the noncommutativity (3.1) does not bring new degrees of freedom on shell. This fact is a general property of this noncommutative deformation. Although we will not give a rigorous proof here, it is not difficult to confirm it, taking into consideration the fact that the noncommutative deformed part in the Lagrangian always contains the parameters $\lambda(\bar{\lambda})$ which are nilpotent.

3.3 Quantum Properties of the noncommutative Wess-Zumino model

We investigate the noncommutative Wess-Zumino model as the quantum field theory on the noncommutative superspace. Since supersymmetry is broken to $\mathcal{N}=0$ in the deformed Wess-Zumino action as already mentioned, we will continue to use the component formalism rather than the superfield formalism to calculate quantum quantities.

We divide the Lagrangian into three part,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\lambda + \mathcal{L}_{\bar{\lambda}}. \tag{3.25}$$

Here $\mathcal{L}_0$ is the original Wess-Zumino Lagrangian as given in (3.17), and the λ-dependent part of the Lagrangian $\mathcal{L}_\lambda$ is

$$\begin{aligned} \mathcal{L}_\lambda \equiv & \frac{g}{2}\lambda^{\mu\alpha}\lambda^{\nu\beta}F(\partial_\mu\psi_\beta)(\partial_\nu\psi_\alpha) + g\lambda^{\mu\nu}F(\partial_\mu F)(\partial_\nu A) \\ & +\frac{g}{3}\lambda^{\mu\nu\rho\sigma}F(\partial_\mu\partial_\nu F)(\partial_\rho\partial_\sigma F), \end{aligned} \tag{3.26}$$

where we have used partial integration and dropped the total derivative terms. $\mathcal{L}_{\bar{\lambda}}$ is the Hermitian conjugate of $\mathcal{L}_\lambda$, $\mathcal{L}_{\bar{\lambda}} = (\mathcal{L}_\lambda)^\dagger$. As seen in Eq.(3.17), we can treat the noncommutative deformed Wess-Zumino model as the theory which is the original Wess-Zumino model and additional vertices with the coupling λ or $\bar{\lambda}$. Note that all couplings in the interaction terms are bosonic, *e.g.*, $\lambda^{\mu\alpha}\lambda^{\nu\beta}$, and the bare fermionic coupling never appear.

We use the standard path-integral formulation. The generating functional is written as follows.

$$Z[J,\eta] = N\exp\left[i\int d^4x\ \mathcal{L}_{\rm int}\left(\delta/i\delta J(x), \delta/\delta\eta(x)\right)\right]\int\mathcal{D}\phi\exp\left[i\int d^4x\left(\mathcal{L}_{\rm free}+\mathcal{L}_{\rm source}\right)\right]. \tag{3.27}$$

Where N is a normalization constant. $\mathcal{L}_{\rm free}$ is the free part of the Lagrangian,

$$\mathcal{L}_{\rm free} = \bar{A}\Box A + i(\partial_\mu\bar{\psi}_{\dot{\alpha}})(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\psi_\beta + \bar{F}F + m(AF - \frac{1}{2}\psi\psi) + \bar{m}(\bar{A}\bar{F} - \frac{1}{2}\bar{\psi}\bar{\psi}), \tag{3.28}$$

and $\mathcal{L}_{\rm int}$ is the interaction part,

$$\begin{aligned}\mathcal{L}_{\rm int} &= \Big[g(A^2F-\psi\psi A) \\ &\quad +\frac{g}{2}\lambda^{\mu\alpha}\lambda^{\nu\beta}F\partial_\mu\psi_\beta\partial_\nu\psi_\alpha+g\lambda^{\mu\nu}F\partial_\mu F\partial_\nu A+\frac{g}{3}\lambda^{\mu\nu\rho\sigma}F\partial_\mu\partial_\nu F\partial_\rho\partial_\sigma F\Big] \\ &\quad +[(\mathrm{h.c.})]\,. \end{aligned} \tag{3.29}$$

$\mathcal{L}_{\rm source}$ is the source terms,

$$\mathcal{L}_{\rm source} = J_A A+\bar{J}_{\bar{A}}\bar{A}+J_F F+\bar{J}_{\bar{F}}\bar{F}+\bar{\eta}^\alpha\psi_\alpha+\eta_{\dot\alpha}\bar{\psi}^{\dot\alpha}. \tag{3.30}$$

The generating functional is rewritten in the following form and we can read off perturbative calculation rules from it.

$$\begin{aligned}Z[J,\eta] &= N'\exp\left[i\int d^4x\mathcal{L}_{\rm int}\left(\delta/i\delta J(x),\delta/\delta\eta(x)\right)\right]\exp\Big[-i\bar{J}_{\bar{A}}\cdot\Delta_{\rm F}J_A \\ &\quad +i\bar{m}J_F\cdot\Delta_{\rm F}J_A+im\bar{J}_{\bar{A}}\cdot\Delta_{\rm F}\bar{J}_{\bar{F}}-iJ_F\cdot\Box\Delta_{\rm F}\bar{J}_{\bar{F}}-\frac{i}{2}m\eta_{\dot\alpha}\cdot\Delta_{\rm F}\eta^{\dot\alpha}-\frac{i}{2}\bar{m}\bar{\eta}^\alpha\cdot\Delta_{\rm F}\bar{\eta}_\alpha \\ &\quad +\bar{\eta}^\alpha\cdot\partial_\mu\Delta_{\rm F}(\sigma^\mu)_{\alpha\dot\alpha}\eta^{\dot\alpha}\Big]\,. \end{aligned} \tag{3.31}$$

Here a dot denotes the integration of spacetime volume, and $\Delta_{\rm F}$ is a propagator,

$$\Delta_{\rm F}=\frac{1}{\Box-|m|^2}. \tag{3.32}$$

To illustrate the quantum property of this theory, we consider vacuum polarization diagrams. It is well-known that divergences of quantum calculation from vacuum diagrams are canceled out in a supersymmetric theory between Bosonic and Fermionic loop corrections, and vacuum energy remains always zero.

In the noncommutative Wess-Zumino model, first nontrivial corrections of the noncommutative deformation to vacuum energy come from two-loop diagrams, at the order $\mathcal{O}(\lambda^2,g^2)$. Fig.3 shows the additional vacuum diagrams. Each diagram is badly ultraviolet divergent since the derivative terms in the vertex added make the convergence of the integral worse. Surprisingly, we have found by explicit calculations that all the contributions from vacuum diagrams at this

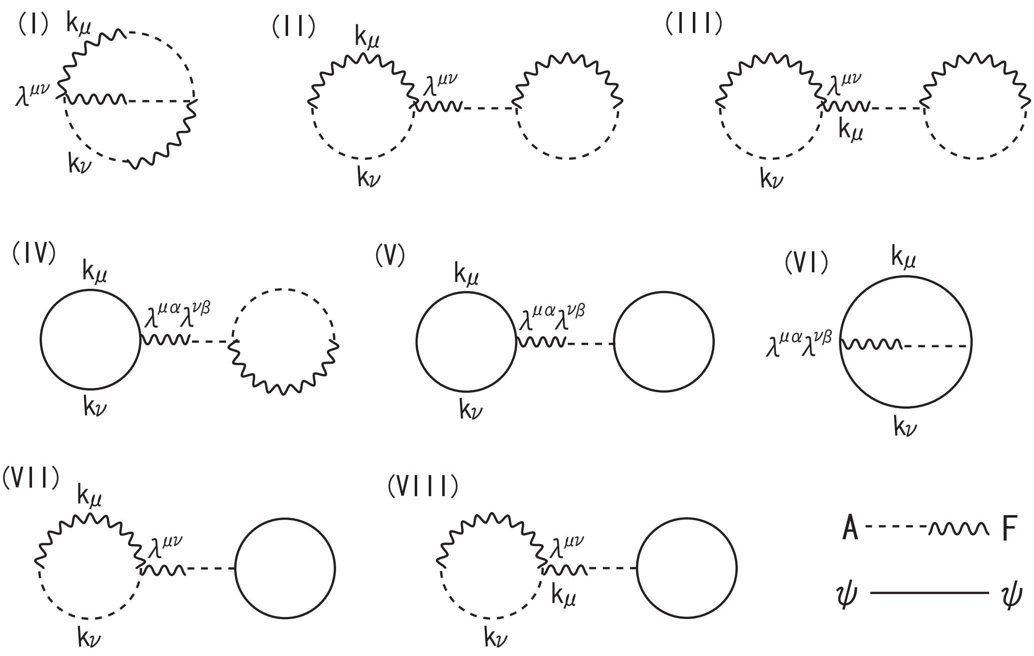

Figure 3: Additional two-loop vacuum diagram at $\mathcal{O}(\lambda^2, g^2)$

order cancel out, even though these diagrams are more divergent than the usual Wess-Zumino model. This is a notable feature of this model. In spite of supersymmetry breaking by the noncommutative deformation, we have no quantum correction to vacuum energy, at least at the nontrivial lowest order of the perturbation.

It is known that the vacuum energy of the ($\mathcal{N} = 1/2$ supersymmetric) Wess-Zumino model in nonanticommutative superspace has no contribution of vacuum loop diagrams at all order, despite the supersymmetry is half broken[26]. The cancellation of the two-loop vacuum diagrams in this model may be related to that.

It should be noted that interaction terms added to $\mathcal{L}_{int}$ are finite thanks to nilpotency of λ. We can split the interaction part of the Lagrangian into λ-dependent $\mathcal{L}^{\lambda}_{\text{int}}$ and λ-independent part $\mathcal{L}^{\lambda=0}_{\text{int}}$. The exponential function of $\mathcal{L}^{\lambda}_{\text{int}}$ in the generating functional (3.27) is expanded in

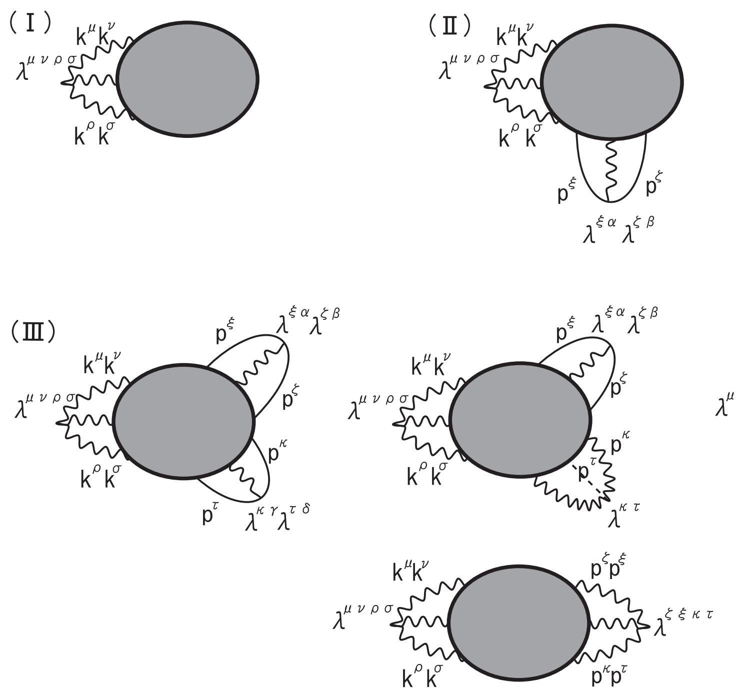

Figure 4: F cubic contribution to Feynman diagram. Diagram (I) is of the order $\mathcal{O}(\lambda^4)$, (II) is $\mathcal{O}(\lambda^6)$ and (III) is $\mathcal{O}(\lambda^8)$.

finite series,

$$
\begin{aligned}
Z[J,\eta] \;=\;& N' \exp\left[i \int d^4x \mathcal{L}_{\text{int}}^{\lambda=0}\left(\delta/\delta J, \delta/\delta\eta\right)\right] \\
& \times \left[1 + \int d^4x \; \mathcal{L}_{\text{int}}^{\lambda}\left(\delta/\delta J, \delta/\delta\eta\right) + (\text{terms up to } \mathcal{O}(\lambda^8, \bar{\lambda}^8))\right] \\
& \times \exp\Bigg[- i\bar{J}_{\bar{A}} \cdot \Delta_{\text{F}} J_A + i\bar{m} J_F \cdot \Delta_{\text{F}} J_A + im \bar{J}_{\bar{A}} \cdot \Delta_{\text{F}} \bar{J}_{\bar{F}} \\
& -iJ_F \cdot \Box \Delta_{\text{F}} \bar{J}_{\bar{F}} - \frac{i}{2} m \eta_{\dot{\alpha}} \cdot \Delta_{\text{F}} \eta^{\dot{\alpha}} - \frac{i}{2} \bar{m} \bar{\eta}^{\alpha} \cdot \Delta_{\text{F}} \bar{\eta}_{\alpha} + \bar{\eta}^{\alpha} \cdot \partial_\mu \Delta_{\text{F}} (\sigma^\mu)_{\alpha\dot{\alpha}} \eta^{\dot{\alpha}} \Bigg] . \quad (3.33)
\end{aligned}
$$

The expansion series of the λ-dependent interaction part is in the second line. 1 in the second line corresponds to the terms in the usual (non-deformed) Wess-Zumino model.

It means in diagrammatic language that only finite sort of diagrams are added under the noncommutative deformation. For example, all the diagrams including at least one F^3 interaction term are classified as in Fig.4, where we omit $\bar{\lambda}$-dependent part for simplicity. The shaded ovals are diagrams which come from $\mathcal{L}_{\text{int}}^{\lambda=0}$, *i.e.* the undeformed Wess-Zumino model.

4 Twisted Symmetry

In this section we explain how to construct a deformed Lie algebra. The noncommutative space that we are interested in is realized as the representations of the deformed Lie algebra. The deformation is systematically achieved in Hopf algebraic way.

Some formulation of noncommutative space with Hopf algebra has been known in the context of *Quantum Group*. For example, *k-deformed Poincaré algebra* provides the space with a type of noncommutative relation Eq.(2.4). We will see that noncommutative space Eq.(1.2) can be derived too, by the deformation of Poincaré algebra, namely *Drinfel'd twist deformation*. The work of Aschieri *et al.*[33], which is an interesting attempt to construct the theory of noncommutative gravity by Drinfel'd Hopf twisted algebra, is a good introduction to the formalism of Hopf algebra method.

To grasp the issue, we consider Poincaré algebra as a specific example of a symmetry Lie algebra. Poincaré algebra $\mathcal{P}$ consists of translation generators P^μ and Lorentz generators $M^{\mu\nu}$, satisfying the commutation relations as follows.

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0, \\
&[M_{\mu\nu}, M_{\rho\sigma}] = i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho}, \\
&[M_{\mu\nu}, P_\rho] = -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu. \qquad (4.1)
\end{aligned}
$$

4.1 Universal Envelope

Hopf algebra is most naturally constructed upon an algebra which posses an associative product and a unit element, although Lie algebra has neither. A popular approach to give a Lie algebra both properties is the *universal envelope* of an algebra. The universal enveloping Lie algebra $\mathcal{U}(\mathcal{G})$ is a natural extension of a Lie algebra $\mathcal{G}$. The product in $\mathcal{U}(\mathcal{G})$ is defined as a formal tensor product. For example $P^\mu \otimes P^\nu$ is the product of P^μ and P^ν. $P^\mu \otimes P^\nu$, $P^\mu \otimes M^{\rho\sigma}$ and $M^{\mu\nu} \otimes P^\rho \otimes P^\sigma$ are products in $\mathcal{U}(\mathcal{P})$. The product is obviously associative but not commutative.

A symbol of the tensor product $\otimes$ in $\mathcal{U}(\mathcal{G})$ can be omitted without confusion, such as $P^\mu P^\nu$, $P^\mu M^{\rho\sigma}$ and $M^{\mu\nu} P^\rho P^\sigma$. In the following, we will omit the symbol of the tensor product in universal enveloping algebra and assume $\otimes$ as the symbol of tensor products in the sense of Hopf algebra.

$\mathcal{U}(\mathcal{G})$ is a linear space over $\mathcal{K}$ spanned by the products of $\mathcal{G}$. Here $\mathcal{K}$ is a base field[4]. When we consider universal enveloping Poincaré algebra $\mathcal{U}(\mathcal{P})$, $\mathcal{K}$ is complex number $\mathbb{C}$. An element of $\mathcal{U}(\mathcal{G})$ is the polynomial of $\mathcal{G}$ with coefficient $\mathcal{K}$.

The product in $\mathcal{U}(\mathcal{G})$ should be consistent with Lie bracket in $\mathcal{G}$. For the compatibility, we impose an exchangeability of a commutator $[X, Y]$ and $XY - YX$ for any $X, Y \in \mathcal{G}$. Which is to say, $XY - YX$ can be replaced with $[X, Y]$ no matter when or where,

$$[X, Y] = XY - YX. \tag{4.2}$$

This formula is nontrivial than it looks. The Lie bracket in LHS results in some linear combination of elements in $\mathcal{G}$, because of closure property of Lie algebra. On the other hand, each term in RHS is not a linear combination of $\mathcal{G}$ but a tensor product of rank two in $\mathcal{U}(\mathcal{G})$. When we take a matrix representation for the Lie algebra, Eq.(4.2) is the *definition* of Lie bracket. Now it is imposed at the level of the algebra in this case, that is independent of a representation.

In mathematics, a universal enveloping algebra is formally defined as the quotient algebra,

$$\mathcal{U}(\mathcal{G}) = \mathcal{T}(\mathcal{G})/\mathcal{I}. \tag{4.3}$$

Here $\mathcal{T}(\mathcal{G})$ is a linear space over $\mathcal{K}$ spanned by the bases of tensor products of elements in $\mathcal{G}$,

$$\mathcal{T}(\mathcal{G}) = \bigoplus_{l=0}^{\infty} \mathcal{T}^l(\mathcal{G}), \tag{4.4}$$

$$\mathcal{T}^0 = \mathcal{K}, \qquad \mathcal{T}^l(\mathcal{G}) = \overbrace{\mathcal{G} \cdots \mathcal{G}}^{l},$$

and $\mathcal{I}$ is an ideal generated by the elements $XY - YX - [X, Y]$ in $\mathcal{U}(\mathcal{G})$.

[4]Usually we think that the base $\mathcal{K}$ is a certain field, complex number $\mathbb{C}$ or real number $\mathbb{R}$ for instance. But when we consider a supersymmetric case later, $\mathcal{K}$ has to be enlarged to a ring to include Grassmann number.

It is obvious from the construction that the representations of a universal enveloping algebra $\mathcal{U}(\mathcal{G})$ are same as those of the original algebra $\mathcal{G}$.

4.2 Hopf algebra

In this section, we will give a brief introduction to Hopf algebra. For details of Hopf algebra and quantum group, we refer to good text books and the references therein [13, 14, 15, 16].

Hopf algebra is an extended structure of an algebra, containing familiar concept of a multiplication and a unit element. *algebra* $(\mathcal{A}, +, \cdot)$ which we assume here has an addition which is denoted by $+$, and a multiplication, denoted by $\cdot$. $\mathcal{A}$ is an Abelian group for $+$, and we denote a unit element of the addition by 0, and an inverse of element a of $\mathcal{A}$ by $-a$. However, for a multiplication, we demand $\mathcal{A}$ only to be a unital semigroup, which need not to be Abelian. So $\mathcal{A}$ has a unit element of the multiplication denoted by 1, but an inverse element of the multiplication for $a \in \mathcal{A}$ is not necessary. An addition and a multiplication on $\mathcal{A}$ are compatible to the following distributive properties,

$$\begin{aligned} a \cdot (b + c) &= a \cdot b + a \cdot c, \\ (a + b) \cdot c &= a \cdot c + b \cdot c, \end{aligned}$$

for all $a, b, c \in \mathcal{A}$. These conditions are almost similar to that of a *ring*. Thus integers $\mathbb{Z}$ and real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ satisfy the conditions. Universal enveloping Lie algebra is also one of the such algebras.

Firstly, we give the general definition of Hopf algebra. Hopf algebra H is a linear space over

$\mathcal{K}$ and has the linear maps on it, $(H, m, i, \Delta, \epsilon, \gamma; \mathcal{K})$. The five linear maps are

$$
\begin{aligned}
\text{product} \quad & m : H \otimes H \to H, \\
\text{unit} \quad & i \quad : \mathcal{K} \to H, \\
\text{coproduct} \quad & \Delta : H \to H \otimes H, \\
\text{counit} \quad & \epsilon \quad : H \to \mathcal{K}, \\
\text{antipode} \quad & \gamma \quad : H \to H.
\end{aligned}
$$

We use H for the linear space on which the maps of Hopf algebra act as well as Hopf algebra itself.

The product m in Hopf algebra is the map version of an ordinary product and satisfies the following relation,

$$
m \circ (\mathrm{id} \otimes m) = m \circ (m \otimes \mathrm{id}). \tag{4.5}
$$

Here id denotes an identity map and $\circ$ stands for composite mapping. Eq.(4.5) is simply the associative condition of the product, $(ab)c = a(bc)$. It works on a product as follows.

$$
\begin{aligned}
(a \cdot b) \cdot c &= m(a \otimes b) \cdot c \\
&= m(m(a \otimes b) \otimes c) \\
&= m \circ (m \otimes \mathrm{id})(a \otimes b \otimes c) \\
&= m \circ (\mathrm{id} \otimes m)(a \otimes b \otimes c) \\
&= m(a \otimes m(b \otimes c)) \\
&= a \cdot (b \cdot c),
\end{aligned} \tag{4.6}
$$

here $a \cdot b \equiv m(a \otimes b)$, and $a, b, c \in H$. This is depicted in Fig.5, which shows that we can get the same result in any path which we choose.

The unit i is the map of giving a unit element of the multiplication in Hopf algebra and

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "m\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes m"'] & H \otimes H \arrow[d, "m"'] \\ H \otimes H \arrow[r, "m"'] & H \end{tikzcd}$$

Figure 5: Property of the product.

$$\begin{tikzcd} \mathcal{K} \otimes H \arrow[r, leftrightarrow, "\cong"'] \arrow[d, "i\otimes\mathrm{id}"'] & H \arrow[r, leftrightarrow, "\cong"'] \arrow[d, "\mathrm{id}"'] & H \otimes \mathcal{K} \arrow[d, "\mathrm{id}\otimes i"'] \\ H \otimes H \arrow[r, "m"'] & H & H \otimes H \arrow[l, "m"] \end{tikzcd}$$

Figure 6: Property of the unit.

satisfies

$$m \circ (i \otimes \mathrm{id}) = \mathrm{id} = m \circ (\mathrm{id} \otimes i), \tag{4.7}$$

which is depicted in Fig.6. $\mathcal{K} \otimes H$ is naturally identified with H, because we are free to distribute the coefficient constant among factors of the tensor product. That is the same for a tensor product of higher order.

$$\mathcal{K} \otimes H \otimes H \otimes \cdots \cong H \otimes \mathcal{K} \otimes H \otimes \cdots \cong H \otimes H \otimes \mathcal{K} \otimes \cdots \cong \cdots \cong H \otimes H \otimes \cdots \tag{4.8}$$

The product and the unit are called *algebra*. Their functions are the same as that of the same name in the original algebra.

On the other hand, a coproduct and a counit are called *Coalgebra*. They are dual operations of *algebra* in a sense.

The coproduct Δ, which is the dual of the product, is the map which splits an element in Hopf algebra, and satisfies

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta. \tag{4.9}$$

Eq.(4.9) is called *coassociative condition* , depicted Fig.7. Eq.(4.9) works on an element $h \in H$

$$\begin{tikzcd} H \otimes H \otimes H & H \otimes H \arrow[l, "\Delta\otimes\mathrm{id}"'] \\ H \otimes H \arrow[u, "\mathrm{id}\otimes\Delta"'] & H \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

Figure 7: Property of the coproduct.

in such a way that

$$\begin{aligned} (\mathrm{id}\otimes\Delta)\circ\Delta(h) &= (\mathrm{id}\otimes\Delta)h_{(1)}\otimes h_{(2)} \\ &= h_{(1)}\otimes h_{(2)(1)}\otimes h_{(2)(1)}, \end{aligned} \tag{4.10}$$

for LHS and

$$\begin{aligned} (\Delta\otimes\mathrm{id})\circ\Delta(h) &= (\mathrm{id}\otimes\Delta)h_{(1)}\otimes h_{(2)} \\ &= h_{(1)(1)}\otimes h_{(1)(2)}\otimes h_{(2)}, \end{aligned} \tag{4.11}$$

for RHS. Here we used Sweedler's notation,

$$\Delta(h) = \sum_i h_1^{(i)} \otimes h_2^{(i)} = h_{(1)} \otimes h_{(2)}, \tag{4.12}$$

where a summation for index i is omitted in the last equation. From these we have the following relation for any $h \in H$,

$$h_{(1)}\otimes h_{(2)(1)}\otimes h_{(2)(1)} = h_{(1)(1)}\otimes h_{(1)(2)}\otimes h_{(2)} \equiv h_{(1)}\otimes h_{(2)}\otimes h_{(3)} \tag{4.13}$$

The counit ϵ, the dual of the unit, satisfies

$$(\mathrm{id}\otimes\epsilon)\circ\Delta = \mathrm{id} = (\epsilon\otimes\mathrm{id})\circ\Delta. \tag{4.14}$$

This condition is depicted in Fig.8. It leads to the equation,

$$\epsilon(h_{(2)})h_{(1)} = \epsilon(h_{(1)})h_{(2)} \;\in H \cong \mathcal{K}\otimes H. \tag{4.15}$$

Fig.5 and Fig.7 make clear the meaning of that the product and coproduct are dual. Let us reverse the directions of arrows, and exchange m for Δ in Fig.5, then it gives the same diagram

$$
\begin{tikzcd}
\mathcal{K} \otimes H \arrow[r, leftrightarrow, "\cong"] & H \arrow[r, leftrightarrow, "\cong"] & H \otimes \mathcal{K} \\
H \otimes H \arrow[u, "\epsilon\otimes\mathrm{id}"'] & H \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[u, "\mathrm{id}"'] & H \otimes H \arrow[u, "\mathrm{id}\otimes\epsilon"']
\end{tikzcd}
$$

Figure 8: Property of the counit.

$$
\begin{tikzcd}
H \arrow[rr, "\epsilon"'] \arrow[d, "\Delta"'] & & \mathcal{K} \arrow[ddd, bend left, "i"] \\
H \otimes H \arrow[r, "\mathrm{id}\otimes\gamma"'] \arrow[d, "\gamma\otimes\mathrm{id}"'] & H \otimes H \arrow[d, "m"'] & \\
H \otimes H \arrow[r, "m"'] & H &
\end{tikzcd}
$$

Figure 9: Property of the antipode.

in Fig.7. Inversely reversing arrows and replace Δ with m in Fig.7, we get Fig.7. We can see that the unit and counit are dual operations in a similar way from Fig.6 and Fig.8.

The antipode γ should satisfy

$$m \circ (\gamma \otimes \mathrm{id}) \circ \Delta = i \otimes \epsilon = m \circ (\mathrm{id} \otimes \gamma) \circ \Delta, \tag{4.16}$$

and is depicted Fig.4.16. The antipode is, roughly speaking, the map which gives the inverse element of the multiplication in the Hopf algebra. But it is not necessary for the antipode to be the inverse operation. For instance we do not demand the condition of $\gamma^2 = \mathrm{id}$.

For compatibility of the algebra and the coalgebra, we require the homomorphisms,

$$\begin{aligned} \Delta(hg) &= \Delta(h)\Delta(g), \\ \epsilon(hg) &= \epsilon(h)\epsilon(g), \end{aligned} \tag{4.17}$$

for all $h, g \in H$, and the conditions for the unit elements

$$\begin{aligned} \Delta(\hat{\mathbf{1}}) &= \hat{\mathbf{1}} \otimes \hat{\mathbf{1}}, \\ \epsilon(\hat{\mathbf{1}}) &= 1. \end{aligned} \tag{4.18}$$

We write a unit element in the Hopf algebra as $\hat{\mathbf{1}}$, to avoid mistaking it for a unit element 1 in $\mathcal{K}$.

On the other hand, the antipode is an *antialgebra* map,

$$\begin{aligned}\gamma(hg) &= \gamma(g)\gamma(h), \\ \gamma(\hat{\mathbf{1}}) &= \hat{\mathbf{1}},\end{aligned} \tag{4.19}$$

and an *anticoalgebra* map, at the same time.

$$\begin{aligned}(\gamma \otimes \gamma) \circ \Delta(h) &= \tau \circ \Delta \circ \gamma(h), \\ \epsilon\gamma(h) &= \epsilon(h).\end{aligned} \tag{4.20}$$

Here we use the *transposition* map $\tau : H \otimes H \to H \otimes H$, which is the linear map with the condition,

$$\tau(a \otimes b) = b \otimes a, \qquad \forall a, b \in H. \tag{4.21}$$

The product of the elements among Hopf algebra is defined as follows.

$$(h \otimes g)(h' \otimes g') = hh' \otimes gg' \tag{4.22}$$

The product for the tensor product of rank three or above is defined in the same way.

Universal enveloping Lie algebra $\mathcal{U}(\mathcal{G})$ can become a Hopf algebra with the definition of the operations on $\mathcal{U}(\mathcal{G})$ as follows.

product	$m(g \otimes h) = gh,$	(4.23)
unit	$i(k) = k\hat{\mathbf{1}},$	(4.24)
coproduct	$\Delta(g) = g \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes g,$	(4.25)
counit	$\epsilon(g) = 0,$	(4.26)
antipode	$\gamma(g) = -g.$	(4.27)

These definitions are all for $g, h \in \mathcal{G}$ and $k \in \mathcal{K}$. The definition of the product is same for $g, h \in \mathcal{U}(\mathcal{G})$. The definition Eq.(4.25)-(4.27) are extended to whole $\mathcal{U}(\mathcal{G})$ with the relations

Eq.(4.17),(4.18) and (4.19). The definition of antipode looks natural because a Lie group can be constructed by exponentiating generators of a Lie algebra, so e^{-h} gives the inverse of e^h.

The coproduct works as the algebra homomorphism surely for $\mathcal{U}(\mathcal{G})$ because from the property of Eq.(4.17)

$$\begin{aligned}\Delta(g)\Delta(h)-\Delta(h)\Delta(g) &= \Delta(gh)-\Delta(hg)\\ &= \Delta(gh-hg)\\ &= \Delta([g,h]),\end{aligned} \tag{4.28}$$

that is consistent with the Lie algebra structure. It can be confirmed for other maps in a similar way.

4.3 Drinfel'd Twist

There is a method developed by Drinfel'd[17] to transform a Hopf algebra into another Hopf algebra systematically.

We consider some Hopf algebra $(H, m, i, \Delta, \epsilon, \gamma; \mathcal{K})$. Then we choose a biproduct element $\mathcal{F} \in H \otimes H$ which is called a *twist element*. The twist element must be invertible, $\exists \mathcal{F}^{-1} \in H$, and satisfy two conditions.

First condition is the *twist equation*;

$$\mathcal{F}_{12}(\Delta_0 \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta_0)\mathcal{F}. \tag{4.29}$$

$\mathcal{F}_{12}$ means that $\mathcal{F}$ acts on the first and second factors of the tensor product,

$$\mathcal{F}_{12} = \mathcal{F} \otimes \mathrm{id} = F_{[1]} \otimes F_{[2]} \otimes \mathrm{id}. \tag{4.30}$$

Here we use Sweedler's notation again for $\mathcal{F}$,

$$\mathcal{F} = \sum_i F_1^{(i)} \otimes F_2^{(i)} = F_{[1]} \otimes F_{[2]}, \tag{4.31}$$

where summation of index i is assumed and the each element is labeled by the number with the brackets [ ] to distinguish the elements from coproduct ones. A similar equation holds for $\mathcal{F}_{23} = \mathrm{id} \otimes \mathcal{F}$. Eq. (4.29) guarantees the *coassociativity* of the twisted Hopf algebra.

Second is the counit condition,

$$(\epsilon \otimes \mathrm{id})\mathcal{F} = \hat{\mathbf{1}} = (\mathrm{id} \otimes \epsilon)\mathcal{F}. \tag{4.32}$$

The Hopf algebra H is deformed by the twist element. The twisted Hopf algebra is redefined only by changing the coproduct and antipode. We define the new coproduct and the antipode as follows.

$$\Delta_t(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \tag{4.33}$$

$$\gamma_t(h) = U\gamma(h)U^{-1}, \tag{4.34}$$

where $U = F_{[1]}\gamma(F_{[2]})$, $U^{-1} = \gamma(\tilde{F}_{[1]})\tilde{F}_{[2]}$, and we defined $\mathcal{F}^{-1} \equiv \tilde{F}_{[1]} \otimes \tilde{F}_{[2]}$.

Then $H'(H, m, i, \Delta_h, \epsilon, \gamma_t; \mathcal{K})$ is also a Hopf algebra. These definitions of new coproduct and antipode clearly keep their original properties, for example,

$$\begin{aligned}
\Delta_t(hg) &= \mathcal{F}\Delta(hg)\mathcal{F}^{-1} \\
&= \mathcal{F}\Delta(h)\Delta(g)\mathcal{F}^{-1} \\
&= \mathcal{F}\Delta(h)\mathcal{F}^{-1}\mathcal{F}\Delta(g)\mathcal{F}^{-1} \\
&= \Delta_t(h)\Delta_t(g),
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
\gamma_t(hg) &= U\gamma(hg)U^{-1} \\
&= U\gamma(g)U^{-1}U\gamma(h)U^{-1} \\
&= \gamma_t(g)\gamma_t(h),
\end{aligned} \tag{4.36}$$

which are same as Eq.(4.17) and Eq.(4.19).

The inverse twist element should satisfy the equation,

$$\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = \hat{\mathbf{1}} \otimes \hat{\mathbf{1}}. \tag{4.37}$$

From this equation and the twist equation (4.29), we see a similar twist equation for the inverse twist element,

$$(\Delta\otimes\mathrm{id})\mathcal{F}^{-1}\mathcal{F}_{12}^{-1}=(\mathrm{id}\otimes\Delta)\mathcal{F}^{-1}\mathcal{F}_{23}^{-1}. \tag{4.38}$$

Consider a special type of a twist element of universal enveloping Lie Hopf algebra $\mathcal{U}(\mathcal{G})$,

$$\mathcal{F}=\exp\left(\sum_i c_{ij}\ h_i\otimes h_j\right). \tag{4.39}$$

Here h_i is an element in Abelian subalgebra of $\mathcal{U}(\mathcal{G})$ and c_{ij} is a coefficient of $\mathcal{K}$. Abelian subalgebra elements all commute with each other, *i.e.*

$$[h_i,h_j]=0, \tag{4.40}$$

for all i,j. Eq.(4.39) is expanded formally in the infinite series,

$$\mathcal{F}=\hat{\mathbf{1}}\otimes\hat{\mathbf{1}}+c_{ij}\ h_i\otimes hj+\frac{1}{2!}(c_{ij}\ h_i\otimes h_j)(c_{lm}\ h_l\otimes h_m)+\cdots. \tag{4.41}$$

The twist element (4.41) obviously satisfies the counit condition Eq.(4.32), since the second term and subsequent terms vanish when they are acted by the counit from Eq.(4.26).

The twist element (4.41) satisfies the twist equation (4.29). We verify it in a straightforward way with the calculations of Hopf algebra.

Proof. From Eq.(4.41) and (4.22), we obtain

$$\begin{aligned}\mathcal{F} &= \exp\left(\sum_i c_{ij}\ h_i\otimes h_j\right)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}c_{i_1j_1}\cdots c_{i_nj_n}(h_{i_1}\otimes h_{j_1})\cdots(h_{i_n}\otimes h_{j_n})\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}c_{i_1j_1}\cdots c_{i_nj_n}(h_{i_1}\cdots h_{i_n})\otimes(h_{j_1}\cdots h_{j_n}).\end{aligned} \tag{4.42}$$

The coproduct acts on the product of h_i in the following way.

$$\begin{aligned}\Delta(h_{i_1}\cdots h_{i_n}) &= \Delta(h_{i_1})\cdots\Delta(h_{i_n})\\ &= \prod_{k=1}^{n}(h_{i_k}\otimes\hat{\mathbf{1}}+\hat{\mathbf{1}}\otimes h_{i_k})\end{aligned} \tag{4.43}$$

We can rearrange freely the order of h's in a factor of the tensor product, because of their commutativity. After an appropriate reassignment of indices, we find

$$\begin{aligned}
\mathcal{F}_{12}(\Delta\otimes\mathrm{id})(\mathcal{F}) &= (F_{[1]}\otimes F_{[2]}\otimes\hat{\mathbf{1}})(\Delta\otimes\mathrm{id})(F_{[1]}\otimes F_{[2]}) \\
&= \left(\sum_{n=0}^{\infty}\frac{1}{n!}c_{i_1j_1}\cdots c_{i_nj_n}\; h_{i_1}\cdots h_{i_n}\otimes h_{j_1}\cdots h_{j_n}\otimes\hat{\mathbf{1}}\right) \\
&\quad\times\left(\sum_{m=0}^{\infty}\frac{1}{m!}c_{i_1j_1}\cdots c_{i_mj_m}\sum_{l=0}^{m}\binom{m}{l}h_{i_1}\cdots h_{i_{l-1}}\otimes h_{i_l}\cdots h_{i_m}\otimes h_{j_1}\cdots h_{j_m}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{m}\frac{1}{n!m!}\binom{m}{l}c_{i_1j_1}\cdots c_{i_nj_n}c_{i'_1j'_1}\cdots c_{i'_mj'_m} \\
&\quad\times\left(h_{i_1}\cdots h_{i_n}h_{i'_1}\cdots h_{i'_{l-1}}\otimes h_{j_1}\cdots h_{j_n}h_{i'_l}\cdots h_{i'_m}\otimes h_{j'_1}\cdots h_{j'_m}\right),
\end{aligned}\tag{4.44}$$

and

$$\begin{aligned}
\mathcal{F}_{23}(\mathrm{id}\otimes\Delta)(\mathcal{F}) &= (\hat{\mathbf{1}}\otimes F_{[1]}\otimes F_{[2]})(\mathrm{id}\otimes\Delta)(F_{[1]}\otimes F_{[2]}) \\
&= \sum_{n'=0}^{\infty}\sum_{m'=0}^{\infty}\sum_{l'=0}^{m'}\frac{1}{n'!m'!}\binom{m'}{l'}c_{i_1j_1}\cdots c_{i_{n'}j_{n'}}c_{i'_1j'_1}\cdots c_{i'_{m'}j'_{m'}} \\
&\quad\times\left(h_{i'_1}\cdots h_{i'_{m'}}\otimes h_{i_1}\cdots h_{i_{n'}}h_{j'_1}\cdots h_{j'_{l'-1}}\otimes h_{j_1}\cdots h_{j_{n'}}h_{j'_{l'}}\cdots h_{j'_{m'}}\right).
\end{aligned}\tag{4.45}$$

Every coefficient c is contracted with two h, which are in the different factor of tensor product. The structure of the contractions is similar to each other and appears again and again in the series. So each term in the expansion in Eq.(4.44) and Eq.(4.45) is characterized by three numbers, α, β and γ. α is the number of the contraction between h in the first and second factors of the tensor product. β and γ is the number of the contraction in the first and third factors, and in the second and third factors respectively.

We can read α, β and γ from Eq.(4.44) and Eq.(4.45).

$$\left\{\begin{array}{lccc}\alpha = & n & = & l' \\ \beta = & l & = & m'-l' \\ \gamma = & m-l & = & n'\end{array}\right. .\tag{4.46}$$

From these relations, both (n,m,l) and (n',m',l') are determined with α, β and γ inversely. All possible combinations of nonnegative integers appears in (α,β,γ). Then corresponding (n,m,l)

and (n', m', l') also run through all nonnegative integers. Moreover the coefficients of the terms of (n, m, l) and of (n', m', l'), which are associated with the numbers (α, β, γ) are identical,

$$\frac{1}{m!n!}\binom{m}{l} = \frac{1}{m'!n'!}\binom{m'}{l'} = \frac{1}{\alpha!\beta!\gamma!}. \tag{4.47}$$

So we have confirmed that the twist equation (4.29) is satisfied order by order, if the twist element is constructed by Eq.(4.39) using the elements in Abelian subalgebra only. □

4.4 Representation Space

An interesting situation is that a Hopf algebra acts on some other structure. Let us consider the case that a Hopf algebra H acts on a vector space over $\mathcal{K}$, $(V, +; \mathcal{K})$. V has an addition which is compatible a scalar multiplication of $\mathcal{K}$,

$$k \cdot (v + w) = k \cdot v + k \cdot w,$$

$$(k + l) \cdot v = k \cdot v + l \cdot v,$$

for all $v, w \in V$ and $k, l \in \mathcal{K}$.

Hopf algebra H acts on V as an *endomorphism* of V.

$$H : V \to V \tag{4.48}$$

If $h \in H$ acts on $v \in V$ from the left side, we denote $h : a \to a'$ as $h \triangleright a = a'$. In this case H is a *left action* of V or V is a *left module* of H. Right action or right module is defined in the same way. V is called also *representation space* of H. The action on V should be compatible with the operation on H such that,

$$g \triangleright (h \triangleright v) = (gh) \triangleright v, \tag{4.49}$$

for all $h, g \in H$ and $v \in V$.

Next we consider the case that V has also a Hopf-algebra-like structure compatible with H. This enable us to write an element v of V with a tensor product, and the multiplication map

$m : V \otimes V \to V$ is defined,

$$m(v \otimes w) = v \cdot w. \tag{4.50}$$

For compatibility with the maps on H, the following conditions are imposed.

$$\begin{aligned} h \triangleright (v \cdot w) &= h \triangleright m(v \otimes w) \\ &= m \circ \Delta(h) \triangleright (v \otimes w) \\ &= m(h_{(1)} \triangleright v \otimes h_{(2)} \triangleright w). \end{aligned} \tag{4.51}$$

(Actually v and w can be in the different representations. We will not discuss about it any further here.)

Let H be a Hopf algebra and let V be a presentation space of H, with the multiplication explained above. Now we consider the twisted Hopf algebra H' of H with the twist element $\mathcal{F}$.

The product in H is not modified in the twist operation. However *the product in representation space should be modified.* We can obtain the proper representation space of H', only to change the definition of the multiplication rule[18].

$$v \cdot w = m(v \otimes w) \to v \star w \equiv m(\mathcal{F}^{-1} \triangleright\ v \otimes w) \tag{4.52}$$

This is the definition of the star product in the representation of the twisted Hopf algebra. Hereafter we will omit the symbol of action, $\triangleright$, when it is obvious.

The product (4.52) is associative.

Proof.

$$
\begin{aligned}
(a \star b) \star c &= m \circ \mathcal{F}^{-1}(a \otimes b) \star c \\
&= m \circ \mathcal{F}^{-1} \big(m \circ \mathcal{F}^{-1}(a \otimes b) \otimes c\big) \\
&= m \circ \mathcal{F}^{-1} \circ (m \otimes \mathrm{id}) \circ (\mathcal{F}^{-1} \otimes \mathrm{id})(a \otimes b \otimes c) \\
&= m \circ (m \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id})(\mathcal{F}^{-1}) \circ (\mathcal{F}^{-1} \otimes \mathrm{id})(a \otimes b \otimes c) \\
&= m \circ (\mathrm{id} \otimes m) \circ (\mathrm{id} \otimes \Delta)(\mathcal{F}^{-1}) \circ (\mathrm{id} \otimes \mathcal{F}^{-1})(a \otimes b \otimes c) \\
&= m \circ \mathcal{F}^{-1} \big(\mathrm{id} \otimes m) \circ (\mathrm{id} \otimes \mathcal{F}^{-1})(a \otimes b \otimes c)\big) \\
&= m \circ \mathcal{F}^{-1}(a \otimes b \star c) \\
&= a \star (b \star c)
\end{aligned}
\tag{4.53}
$$

Here in the forth line we used Eq.(4.51), and in the fifth line we used product associativity Eq.(4.5) and the inverse twist equation Eq.(4.38). □

This star product is compatible with the twisted symmetry.

$$
\begin{aligned}
h(a \star b) &= h \circ m \circ \mathcal{F}^{-1}(a \otimes b) \\
&= m \circ \Delta(h) \circ \mathcal{F}^{-1}(a \otimes b) \\
&= m \circ \mathcal{F}^{-1} \circ \mathcal{F} \circ \Delta(h) \circ \mathcal{F}^{-1}(a \otimes b) \\
&= m(\mathcal{F}^{-1} \Delta_t(h) a \otimes b) \\
&= a' \star b',
\end{aligned}
\tag{4.54}
$$

for all $h \in H$ and $a, b \in V$. Here we defined

$$
\begin{cases} a' = h_{t(1)} a \\ b' = h_{t(2)} b \end{cases}, \qquad \Delta_t(h) = h_{t(1)} \otimes h_{t(2)}, \tag{4.55}
$$

and used the definition of the twisted coproduct (4.33). The action on a single element $a \in V$ is not modified, but the action on a product in the representation space V is modified by the twist.

4.5 Twisted Poincaré Algebra

We are now ready to construct the *twisted Poincaré algebra.*

Poincaré algebra can become a Hopf algebra, through universal enveloping, with the definitions of the operations which are already mentioned in section 4.2. Universal enveloping Poincaré Hopf algebra $\mathcal{U}(\mathcal{P})$ is considered as the original symmetry algebra.

We take the coordinate representation as the representation space. Generators of Poincaré algebra are represented as the differential operators on it,

$$\begin{aligned} P_\mu &= i\partial_\mu, \\ M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu). \end{aligned} \tag{4.56}$$

Spacetime coordinate representation allows the Hopf-algebra-like structure into it. Ordinary(non-twisted) product on the spacetime representation is defined as follows.

$$x^\mu \cdot x^\nu \equiv m(x^\mu \otimes x^\nu). \tag{4.57}$$

It is to be noted that the LHS in the above equation is a noncommutative product rather than a commutative object $x^\mu x^\nu \in \mathbb{R}$. It is not surprising since the tensor product in the RHS should not commute. To retain usual commutative coordinate description, we have to impose an equivalence relation such that $x^\mu \cdot x^\nu - x^\nu \cdot x^\mu = 0$. In mathematics, it means that commutative space is recovered as the quotient space with the ideal which is generated by $x^\mu \cdot x^\nu - x^\nu \cdot x^\mu$. In addition to that, to obtain a usual commutative product from the dotted product, we have to assign them to commutative products in some ordering. For instance, in symmetric (Weyl) ordering we impose the relation,

$$\frac{1}{2}(x^\mu \cdot x^\nu + x^\nu \cdot x^\mu) \longleftrightarrow x^\mu x^\nu. \tag{4.58}$$

If we keep in mind these facts, there is no difference between both descriptions. For example, let us see how the Lorentz generator transform the product on both space. Usually,

$$
\begin{aligned}
M_{\mu\nu}(x_\rho x_\sigma) &= \left(i(x_\mu\partial_\nu - x_\nu\partial_\mu)\right)(x_\rho x_\sigma) \\
&= ix_\mu\left((\partial_\nu x_\rho)x_\sigma + x_\rho(\partial_\nu x_\sigma)\right) - ix_\nu\left((\partial_\mu x_\rho)x_\sigma + x_\rho(\partial_\mu x_\sigma)\right) \\
&= i\eta_{\nu\rho}x_\mu x_\sigma - i\eta_{\nu\sigma}x_\mu x_\rho - i\eta_{\mu\rho}x_\nu x_\sigma + i\eta_{\mu\sigma}x_\nu x_\rho. \qquad (4.59)
\end{aligned}
$$

It is just like the behavior of rank two tensor under the Lorentz transformation.

On the other hand, in Hopf algebra,

$$
\begin{aligned}
M_{\mu\nu}x_{\{\rho}\cdot x_{\sigma\}} &= M_{\mu\nu}\, m\left(x_{\{\rho}\otimes x_{\sigma\}}\right) \\
&= m\left(\Delta(M_{\mu\nu})x_{\{\rho}\otimes x_{\sigma\}}\right) \\
&= m\left((M_{\mu\nu}\otimes \hat{\mathbf{1}} + \hat{\mathbf{1}}\otimes M_{\mu\nu})x_{\{\rho}\otimes x_{\sigma\}}\right) \\
&= m\left(M_{\mu\nu}x_{\{\rho}\otimes x_{\sigma\}} + x_{\{\rho}\otimes M_{\mu\nu}x_{\sigma\}}\right) \\
&= m\left(i(x_\mu\partial_\nu - x_\nu\partial_\mu)x_{\{\rho}\otimes x_{\sigma\}} + x_{\{\rho}\otimes i(x_\mu\partial_\nu - x_\nu\partial_\mu)x_{\sigma\}}\right) \\
&= m\left(i\eta_{\nu\rho}x_{\{\mu}\otimes x_{\sigma\}} - i\eta_{\nu\sigma}x_{\{\rho}\otimes x_{\mu\}}\right. \\
&\qquad \left. - i\eta_{\mu\rho}x_{\{\nu}\otimes x_{\sigma\}} + i\eta_{\mu\sigma}x_{\{\nu}\otimes x_{\rho\}}\right) \\
&= i\eta_{\nu\rho}x_{\{\mu}\cdot x_{\sigma\}} - i\eta_{\nu\sigma}x_{\{\rho}\cdot x_{\mu\}} - i\eta_{\mu\rho}x_{\{\nu}\cdot x_{\sigma\}} + i\eta_{\mu\sigma}x_{\{\nu}\cdot x_{\rho\}}
\end{aligned}
\qquad (4.60)
$$

Here braces $\{\}$ denotes the symmetrization, $x_{\{\mu}\otimes x_{\nu\}} = \frac{1}{2}(x_\mu\otimes x_\nu + x_\nu\otimes x_\mu)$. In the column of each tensor product, we can calculate in the same way as an ordinary product and a differential operator. The product $x_{\{\mu}\otimes x_{\nu\}}$ is exactly transformed like as the rank two tensor.

From Eq.(4.60), we can see that the coproduct is defined so as to work as the *Leibnitz rule.* Since the Drinfel'd twist deforms the coproduct, that is just the deformation of the differential structure of the algebra.

Then we choose a twist element as follows.

$$
\mathcal{F}^{PP} = \exp\left(\frac{i}{2}\Theta^{\mu\nu}P_\mu\otimes P_\nu\right),. \qquad (4.61)
$$

where $\Theta^{\mu\nu}$ is a (real) constant with spacetime indices. This twist element satisfies the twist equation(4.29) and the counit condition Eq.(4.32) automatically, because translation generators P^μ are in an Abelian subsector of Poincaré algebra.

In coordinate representation, the twist element $\mathcal{F}^{PP}$ looks like the Moyal product itself.

$$\mathcal{F}^{PP} = \exp\left(-\frac{i}{2}\Theta^{\mu\nu}\partial_\mu \otimes \partial_\nu\right). \tag{4.62}$$

Actually, the formulation in Hopf algebraic method with $\mathcal{F}^{PP}$ produces the same results by the formulation with the Moyal product (2.25).

Consequently the product on the representation space is modified as mentioned previously.

$$\begin{array}{ccc} x^\mu \cdot x^\nu & & x^\mu \star x^\nu \\ = m(x^\mu \otimes x^\nu) & \longrightarrow & = m((\mathcal{F}^{PP})^{-1} x^\mu \otimes x^\nu) \equiv m_t(x^\mu \otimes x^\nu) \end{array} \tag{4.63}$$

We derive all the calculations on the representation space with this star product.

$$\begin{aligned} x^\mu \star x^\nu &= m_t(x^\mu \otimes x^\nu) \\ &= m((\mathcal{F}^{PP})^{-1} x^\mu \otimes x^\nu) \\ &= m\left\{\exp\left(-\frac{i}{2}\Theta^{\rho\sigma} P_\rho \otimes P_\sigma\right)(x^\mu \otimes x^\nu)\right\} \\ &= m\left\{\exp\left(\frac{i}{2}\Theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma\right)(x^\mu \otimes x^\nu)\right\} \\ &= m\left(x^\mu \otimes x^\nu + \frac{i}{2}\Theta^{\rho\sigma}\delta^\mu_\rho \otimes \delta^\nu_\sigma\right) \\ &= x^\mu \cdot x^\nu + \frac{i}{2}\Theta^{\mu\nu} \end{aligned} \tag{4.64}$$

A commutator is calculated with this star product. Then we have the desired noncommutative relation,

$$\begin{aligned} [x^\mu, x^\nu]_\star &\equiv x^\mu \star x^\nu - x^\nu \star x^\mu \\ &= i\Theta^{\mu\nu} \neq 0. \end{aligned} \tag{4.65}$$

It should be mentioned that no *a priori* reason to determine the value of $\Theta^{\mu\nu}$ at the level of the twist operation on the Hopf algebra. Any constant $\Theta^{\mu\nu}$, even if it is not real, will satisfy the twist equation. We fix them to keep the consistency of the algebraic structure in the representation space. The indices should be antisymmetric in μ and ν in the above equation, and $\Theta^{\mu\nu}$ should be real from the Hermiticity condition.

This construction of the noncommutative spacetime was done by Chaichian *et. al* and Oeckl[5, 6]. It is obvious from their construction that the theory has a continuous commutative limit. If we take the limit $\Theta^{\mu\nu} \to 0$, twisted Poincaré algebra reduces to Poincaré algebra, and noncommutative space reduces to commutative space smoothly.

4.6 Twisted Lorentz symmetry

We verify the algebraic consistency of twisted Poincaré algebra. The structure of Poincaré algebra and the coproduct of P_μ are not modified by the twist of $\mathcal{F}^{PP}$, but the coproduct of $M_{\mu\nu}$ is changed,

$$\begin{aligned} \Delta_t^{PP}(M_{\mu\nu}) &= M_{\mu\nu} \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes M_{\mu\nu} \\ &\quad -\frac{1}{2}\Theta^{\rho\sigma}\left[(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \otimes P_\sigma + P_\rho \otimes (\eta_{\sigma\mu}P_\nu - \eta_{\sigma\nu}P_\mu)\right]. \end{aligned} \tag{4.66}$$

If noncommutativity parameter $\Theta^{\mu\nu}$ behave as a constant in the twisted symmetry, the Lorentz generator $M_{\mu\nu}$ annihilate $\Theta^{\mu\nu}$, $M_{\mu\nu}(\Theta^{\rho\sigma}) = 0$. For consistency, $M_{\mu\nu}$ should annihilate $[x^\rho, x^\sigma]_\star = x^\rho \star x^\sigma - x^\sigma \star x^\rho (= \Theta^{\rho\sigma})$. That is confirmed as follows.

$$\begin{aligned} M_{\mu\nu}[x^\rho, x^\sigma]_\star &= M_{\mu\nu} \circ m_t(x^\rho \otimes x^\sigma - x^\sigma \otimes x^\rho) \\ &= M_{\mu\nu} \circ m\left(\left(\mathcal{F}^{\mathcal{P}\mathcal{P}}\right)^{-1} x^\rho \otimes x^\sigma - x^\sigma \otimes x^\rho\right) \\ &= m\left(\left(\mathcal{F}^{\mathcal{P}\mathcal{P}}\right)^{-1} \Delta_t(M_{\mu\nu})\, x^\rho \otimes x^\sigma - x^\sigma \otimes x^\rho\right) \\ &= m\Big[\left(\mathcal{F}^{\mathcal{P}\mathcal{P}}\right)^{-1}\Big\{(ix_\mu\partial_\nu - ix_\nu\partial_\mu) \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes (ix_\mu\partial_\nu - ix_\nu\partial_\mu) \\ &\qquad +\frac{1}{2}\Theta^{\mu'\nu'}(\eta_{\mu'\mu}\partial_\nu - \eta_{\mu'\nu}\partial_\mu) \otimes \partial_{\nu'} + \partial_{\mu'} \otimes (\eta_{\nu'\mu}\partial_\nu - \eta_{\nu'\nu}\partial_\mu)\Big\} \\ &\qquad x^\rho \otimes x^\sigma - x^\sigma \otimes x^\rho\Big] \\ &= 0. \end{aligned} \tag{4.67}$$

Therefore there is no inconsistency in the way of the transformation between LHS and RHS in noncommutative relation (4.65), and $\Theta^{\mu\nu}$ behave as a constant under the twisted Lorentz transformation.

That is a little bit curious because we are inclined to think that the noncommutative parameter may transform in a covariant way under the Lorentz transformation $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$, since $x^{\mu} \star x^{\nu} - x^{\nu} \star x^{\mu}$ looks like a rank two tensor.

$$\begin{aligned}
\Theta^{\mu\nu} &= x^{\mu} \star x^{\nu} - x^{\nu} \star x^{\mu} \longrightarrow \\
\Theta'^{\mu\nu} &= \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} \left(x^{\mu'} \star x^{\nu'} - x^{\nu'} \star x^{\mu'} \right) \\
&= \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} \Theta^{\mu'\nu'},
\end{aligned} \tag{4.68}$$

where $\Lambda^{\mu}{}_{\nu}$ is a representation matrix of the Lorentz transformation group. In the last equation indicate $\Theta^{\mu\nu}$ is transformed as a tensor.

That is not the case with twisted symmetry transformation. $\Theta^{\mu\nu}$ is not a tensor, but a constant, thus it is *invariant* under the twisted Lorentz transformation indeed. Hence the twisted Lorentz transformation is associated with a local transformation or Particle Lorentz transformation.

Next we see the transformation of $x_{\{\mu} \star x_{\nu\}}$, which is a *rank two tensor in noncommutative space.*

$$\begin{aligned}
M_{\mu\nu} x_{\{\rho} \star x_{\sigma\}} &= M_{\mu\nu} \circ m_t (x_{\{\rho} \otimes x_{\sigma\}}) \\
&= M_{\mu\nu} \circ m \left(\left(\mathcal{F}^{\mathcal{P}\mathcal{P}}\right)^{-1} x_{\{\rho} \otimes x_{\sigma\}} \right) \\
&= m \left(\left(\mathcal{F}^{\mathcal{P}\mathcal{P}}\right)^{-1} \Delta_t(M_{\mu\nu})\ x_{\{\rho} \otimes x_{\sigma\}} \right) \\
&= m \left(\left(\mathcal{F}^{\mathcal{P}\mathcal{P}}\right)^{-1} i\eta_{\nu\rho} x_{\{\mu} \otimes x_{\sigma\}} - i\eta_{\nu\sigma} x_{\{\rho} \otimes x_{\mu\}} - i\eta_{\mu\rho} x_{\{\nu} \otimes x_{\sigma\}} + i\eta_{\mu\sigma} x_{\{\nu} \otimes x_{\rho\}} \right) \\
&= i\eta_{\nu\rho} x_{\{\mu} \star x_{\sigma\}} - i\eta_{\nu\sigma} x_{\{\rho} \star x_{\mu\}} - i\eta_{\mu\rho} x_{\{\nu} \star x_{\sigma\}} + i\eta_{\mu\sigma} x_{\{\nu} \star x_{\rho\}}
\end{aligned} \tag{4.69}$$

Eq.(4.69) is quite a similar to Eq.(4.60), if we replace a dotted product with a star product. $x_{\{\mu} \star x_{\nu\}}$ is transformed as a twisted Lorentz tensor exactly.

The other way of the calculation is that the star product is expanded first, and we write the equation in terms of noncommutativity parameter $\Theta^{\mu\nu}$ explicitly. After that, Lorentz generator act on the equation as the ordinary derivative operator, we can derive the same results like as in Eq.(4.67) and Eq.(4.69), after the explicit expansion of the star product. In a sense, we can exchange Lorentz transformation and taking the star product in twisted Hopf algebra. If we want only the results of the calculation, we need not use the twisted Hopf algebraic way instead of a regular procedure. From this aspect, the twisted Hopf algebraic calculation is a changing the point of view. The significant feature of the formulation in a twisted Hopf algebra is that we do not always have to write the noncommutativity parameter $\Theta^{\mu\nu}$ explicitly, in Eq.(4.67) or Eq.(4.69). They are hidden behind the star product and the coproduct. Thus the algebraic structure is exactly same with the non-twisted case, *i.e.*, the case in commutative spacetime, as we can see from a comparison Eq.(4.69) with Eq.(4.60).

In twisted Hopf algebraic way, we can calculate equations so as to maintain all the algebraic structure, even in noncommutative spacetime. That means, in short, the twisted algebra is the symmetry of the noncommutative theory.

5 Extension to Supersymmetric Theory

5.1 $\mathbf{Z}_2$ Graded Hopf Algebra

It is almost straightforward to extend the construction of twisted symmetry for a supersymmetric case. However some preparations are needed because supersymmetric algebra which we want to modify contains the generators of the fermionic nature. To treat these fermionic ingredients, we have to slightly change the definition of a multiplication rule in the Hopf algebra. And we also allow the base $\mathcal{K}$ to be not only a field but a *ring* in the case of a Grassmann number noncommutativity parameters.

We define a $\mathbf{Z}_2$ graded Hopf algebra, in which the multiplication is modified as follows.

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd), \tag{5.1}$$

for $a, c, b, d \in H$. Here $|a|$ denote the "number of fermion" of a,

$$|a| = \begin{cases} 0 & \text{if } a \text{ is fermionic} \\ 1 & \text{if } a \text{ is bosonic} \end{cases} . \tag{5.2}$$

The definition is extended to the rank of more than two, so as to change a sign at every jump over a fermionic element each other. For instance, a product of three columns is given by

$$(a \otimes b \otimes c)(d \otimes e \otimes f) = (-1)^{|c|(|d|+|e|)+|b||d|}(ad \otimes be \otimes cf). \tag{5.3}$$

When the base $\mathcal{K}$ is *Grassmann number ring*, the fermionic numbers should also be consistent with the definition of the $\mathbf{Z}_2$ graded Hopf algebra H. For consistency we impose the anticommutative property of a fermionic number $\lambda \in \mathcal{K}$ with a fermionic element $h_i \in H$, for example,

$$\begin{aligned} \lambda h_1 \otimes h_2 \otimes h_3 &= (-1)^{|\lambda||h_1|} h_1 \lambda \otimes h_2 \otimes h_3 \\ &= (-1)^{|\lambda||h_1|} h_1 \otimes \lambda h_2 \otimes h_3 \\ &= (-1)^{|\lambda|(|h_1|+|h_2|)} h_1 \otimes h_2 \otimes \lambda h_3. \end{aligned} \tag{5.4}$$

5.2 Twisted Super Poincaré Algebra

Instead of Poincaré algebra, we start with super Poincaré algebra, to obtain the non(anti)commutative superspace. Such noncommutative deformation can expect to be the twisted supersymmetric and the twisted Lorentz symmetric deformation. Another approach to construct a theory which maintains Lorentz symmetry and supersymmetry on noncommutative superspace by the spinor formalism is [35].

In addition to Poincaré algebra, $\mathcal{N}=1$ Super Poincaré algebra $\mathcal{SP}$ consists of supercharge Q_α and antisupercharge $\bar{Q}_{\dot\alpha}$. The commutation relations are as follows.

$$\begin{aligned}
&[P_\mu, P_\nu] = 0, \\
&[M_{\mu\nu}, M_{\rho\sigma}] = i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho}, \\
&[M_{\mu\nu}, P_\rho] = -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu, \\
&[P_\mu, Q^\alpha] = 0, \quad [P_\mu, \bar{Q}_{\dot\alpha}] = 0, \\
&[M_{\mu\nu}, Q_\alpha] = i\left(\sigma_{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta, \quad [M_{\mu\nu}, \bar{Q}^{\dot\alpha}] = i\left(\bar\sigma_{\mu\nu}\right)^{\dot\alpha}_{\ \dot\beta}\bar{Q}^{\dot\beta}, \\
&\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu.
\end{aligned} \tag{5.5}$$

In superspace coordinate representation, the generators in $\mathcal{SP}$ are represented as the differential operators on superspace,

$$\begin{aligned}
P_\mu &= i\partial_\mu, \\
M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) - i\theta^\alpha\left(\sigma_{\mu\nu}\right)_\alpha^{\ \beta}\frac{\partial}{\partial\theta^\beta} - i\bar\theta_{\dot\alpha}\left(\bar\sigma_{\mu\nu}\right)^{\dot\alpha}_{\ \dot\beta}\frac{\partial}{\partial\bar\theta_{\dot\beta}}, \\
Q_\alpha &= i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu, \\
\bar{Q}_{\dot\alpha} &= -i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu.
\end{aligned} \tag{5.6}$$

Universal enveloping super Poincaré algebra $\mathcal{U}(\mathcal{SP})$ becomes a $\mathbf{Z}_2$ graded Hopf algebra over

$\mathcal{K}$ by the following definition in the same way as in $\mathcal{U}(\mathcal{G})$.

$$\text{product}: \quad m(X \otimes Y) = XY, \tag{5.7}$$

$$\text{unit}: \quad i(k) = k\hat{\mathbf{1}}, \tag{5.8}$$

$$\begin{aligned} \text{coproduct}: \quad & \Delta(X) = X \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes X, \\ & \Delta(\hat{\mathbf{1}}) = \hat{\mathbf{1}} \otimes \hat{\mathbf{1}}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} \text{counit}: \quad & \epsilon(X) = 0, \\ & \epsilon(\hat{\mathbf{1}}) = 1, \end{aligned} \tag{5.10}$$

$$\begin{aligned} \text{antipode}: \quad & \gamma(X) = -X, \\ & \gamma(\hat{\mathbf{1}}) = \hat{\mathbf{1}}, \end{aligned} \tag{5.11}$$

for $X, Y \in \mathcal{SP}$ and $k \in \mathcal{K}$. These definitions are extended to whole $\mathcal{U}(\mathcal{SP})$ recursively with Eq.(4.17)-(4.19).

The twist element which satisfies the twist equation (4.29) and counit condition Eq.(4.32) is easily constructed from the elements of the Abelian subalgebra, as it is proved in the previous section for bosonic generators. Actually the Abelian subalgebra can include not only bosonic but fermionic generators. That means we can loose the condition such that two generators anticommute with each other if both of them are fermionic, otherwise commute with each other.

We can prove that statement as follows.

Proof. Let us define the following twist element.

$$\mathcal{F} = \exp\left(c^{ij}\ G_i \otimes G'_j\right), \tag{5.12}$$

where c^{ij} is a constant, and G and G' are generators. The constant and generator can be either bosonic or fermionic. The counit condition is clearly satisfied for $\mathcal{F}$, so it is sufficient to check the twist equation. The LHS of the twist equation is,

$$
\begin{aligned}
&\mathcal{F}_{12}(\Delta_0 \otimes \mathrm{id})\mathcal{F} \\
&\quad = \exp\left(c^{ij}\ G_i \otimes G'_j \otimes \hat{\mathbf{1}}\right) \exp\left(c^{ij}\ G_i \otimes \hat{\mathbf{1}} \otimes G'_j + \hat{\mathbf{1}} \otimes G_i \otimes G'_j\right)
\end{aligned}
\tag{5.13}
$$

An explicit calculation shows that arguments of the exponential function in the above equation commute. In the calculations, we have to exercise caution with sign flips.

$$
\begin{aligned}
& \left[c^{ij}\ G_i \otimes G'_j \otimes \hat{\mathbf{1}}, c^{kl}(G_k \otimes \hat{\mathbf{1}} \otimes G'_l + \hat{\mathbf{1}} \otimes G_k \otimes G'_l)\right] \\
= \;& c^{ij}c^{kl}\left\{(-1)^{|c^{kl}|(|G_i|+|G'_j|)+|G'_j||G_k|} G_iG_k \otimes G'_j \otimes G'_l \right.\\
& \left. +(-1)^{|c^{kl}|(|G_i|+|G'_j|)} G_i \otimes G'_jG_k \otimes G'_l \right\} \\
- \;& c^{kl}c^{ij}\left\{(-1)^{|c^{ij}|(|G'_l|+|G_k|)+(|G_i|+|G'_j|)|G'_l|} G_kG_i \otimes G'_j \otimes G'_l \right.\\
& \left. +(-1)^{(|c^{ij}|+|G_i|)(|G_k|+|G'_l|)+|G'_j||G'_l|} G_i \otimes G_kG'_j \otimes G'_l \right\} \\
= \;& c^{ij}c^{kl}\left\{\left((-1)^{|c^{kl}|(|G_i|+|G'_j|)+|G'_j||G_k|} \right.\right.\\
& \left. \quad -(-1)^{|c^{ij}|(|G'_l|+|G_k|+|c^{kl}|)+(|G_i|+|G'_j|)|G'_l|+|G_i||G_k|}\right) \times G_iG_k \otimes G'_j \otimes G'_l \\
+ \;& \left((-1)^{|c^{kl}|(|G_i|+|G'_j|)} - (-1)^{(|c^{ij}|+|G_i|)(|G_k|+|G'_l|)+|G'_j|(|G'_l|+|G_k|)+|c^{ij}||c^{kl}|}\right) \\
& \left. \times G_i \otimes G'_jG_k \otimes G'_l \right\} \\
= \;& 0.
\end{aligned}
\tag{5.14}
$$

Do not sum over the same upper and lower index here. In the last equation, we have used the following fact.

$$
(-1)^a - (-1)^b = \begin{cases} 0 & \text{if } a+b \text{ is even.} \\ -2 \text{ or } 2 & \text{if } a+b \text{ is odd.} \end{cases} \qquad a, b \in \mathbb{Z}
\tag{5.15}
$$

For the sums of indices of the (-1) are

$$
\begin{aligned}
& |c^{kl}|(|G_i| + |G'_j|) + |G'_j||G_k| \\
& \quad +|c^{ij}|(|G'_l| + |G_k| + |c^{kl}|) + (|G_i| + |G'_j|)|G'_l| + |G_i||G_k| \\
& \quad = (|c^{ij}| + |G_i| + |G'_j|)(|c^{kl}| + |G_k| + |G'_l|),
\end{aligned}
\tag{5.16}
$$

and

$$
\begin{aligned}
&|c^{kl}|(|G_i| + |G'_j|) \\
&+(|c^{ij}| + |G_i|)(|G_k| + |G'_l|) + |G'_j|(|G'_l| + |G_k|) + |c^{ij}||c^{kl}| \\
&= (|c^{ij}| + |G_i| + |G'_j|)(|c^{kl}| + |G_k| + |G'_l|).
\end{aligned} \tag{5.17}
$$

The total of the fermionic character $|c^{ij}| + |G_i| + |G'_j|$ should be even, since the argument of exponential function is bosonic. Therefore we can get together the arguments of the exponential functions in Eq.(5.13),

$$
\begin{aligned}
\mathcal{F}_{12}(\Delta_0 \otimes \mathrm{id})\mathcal{F} &= \exp\left(c^{ij}\ G_i \otimes G'_j \otimes \hat{\mathbf{1}}\right) \exp\left(c^{ij}(G_i \otimes \hat{\mathbf{1}} \otimes G'_j + \hat{\mathbf{1}} \otimes G_i \otimes G'_j)\right) \\
&= \exp\left(c^{ij}(G_i \otimes G'_j \otimes \hat{\mathbf{1}} + G_i \otimes \hat{\mathbf{1}} \otimes G'_j + \hat{\mathbf{1}} \otimes G_i \otimes G'_j)\right).
\end{aligned} \tag{5.18}
$$

The RHS of the twist equation is calculated in a similar way.

$$
\begin{aligned}
\mathcal{F}_{23}(\mathrm{id} \otimes \Delta_0)\mathcal{F} &= \exp\left(c^{ij}\ \hat{\mathbf{1}} \otimes G_i \otimes G'_j\right) \exp\left(c^{ij}(G_i \otimes G'_j \otimes \hat{\mathbf{1}} + G_i \otimes \hat{\mathbf{1}} \otimes G'_j)\right) \\
&= \exp\left(c^{ij}(\hat{\mathbf{1}} \otimes G_i \otimes G'_j + G_i \otimes G'_j \otimes \hat{\mathbf{1}} + G_i \otimes \hat{\mathbf{1}} \otimes G'_j)\right)
\end{aligned} \tag{5.19}
$$

Now we have proved that the twist element (5.12) satisfies the conditions, for all the constants c^{ij} and generators G_i in Abelian subalgebra, irrespective of whether each of them is bosonic or fermionic. □

In super Poincaré algebra, an Abelian subalgebra is made up of translation generators P^μ and supercharges Q^α or alternatively, P^μ and anti-supercharges $\bar{Q}^{\dot{\alpha}}$. We cannot choose both Q^α and $\bar{Q}^{\dot{\alpha}}$ because they do not anticommute with each other.

We will consider several twist separately.

5.2.1 P-P Twist

Since super Poincaré algebra is an extension of Poincaré algebra, in other words, super Poincaré algebra contains Poincaré algebra as its subalgebra, the $\mathcal{F}^{PP}$ twist element works well even in superspace as well as ordinary spacetime,

$$\mathcal{F}^{PP} = \exp\left(\frac{i}{2}\Theta^{\mu\nu}P_\mu \otimes P_\nu\right). \tag{5.20}$$

This twist element gives the following noncommutativities,

$$\begin{aligned}
[x^\mu, x^\nu]_\star &= i\Theta^{\mu\nu}, \\
\{\theta^\alpha, \theta^\beta\}_\star &= 0, \\
[x^\mu, \theta^\alpha]_\star &= 0.
\end{aligned} \tag{5.21}$$

We omitted $\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\}_\star = 0$, $\{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\}_\star = 0$ and $[x^\mu, \bar{\theta}^{\dot{\beta}}]_\star = 0$. Hereafter we assume that these commutators, which include the antichiral part of fermionic coordinate $\bar{\theta}^{\dot{\alpha}}$ in superspace, remain always zero unless otherwise noted.

5.2.2 Q-Q Twist

Modifications of the commutator for fermionic coordinates in superspace is achieved by the twist element of Q^α. Consider the following twist element.

$$\mathcal{F}^{QQ} = \exp\left(-\frac{1}{2}C^{\alpha\beta}Q_\alpha \otimes Q_\beta\right), \tag{5.22}$$

where $C^{\alpha\beta}$ is a constant. The products of coordinates with this twist element are

$$\begin{aligned}
\theta^\alpha \star \theta^\beta &= m_t(\theta^\alpha \otimes \theta^\beta) \\
&= m \circ e^{\frac{1}{2}C^{\gamma\delta}Q_\gamma \otimes Q_\delta}(\theta^\alpha \otimes \theta^\beta) \\
&= m\left(\exp\left(\frac{1}{2}C^{\gamma\delta}(i\frac{\partial}{\partial\theta^\gamma} - \sigma^\mu_{\gamma\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}\partial_\mu) \otimes (i\frac{\partial}{\partial\theta^\delta} - \sigma^\nu_{\delta\dot{\delta}}\bar{\theta}^{\dot{\delta}}\partial_\nu)\right)\theta^\alpha \otimes \theta^\beta\right) \\
&= m\left(\theta^\alpha \otimes \theta^\beta + \frac{1}{2}C^{\gamma\delta}\delta^\alpha{}_\gamma \delta^\beta{}_\delta\right) \\
&= \theta^\alpha \cdot \theta^\beta + \frac{1}{2}C^{\alpha\beta},
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
x^\mu \star x^\nu &= m \circ e^{\frac{1}{2}C^{\gamma\delta}Q_\gamma \otimes Q_\delta}(x^\mu \otimes x^\nu) \\
&= m\left(x^\mu \otimes x^\nu + \frac{1}{2}C^{\gamma\delta}(-\sigma^\rho_{\gamma\dot{\gamma}})\bar{\theta}^{\dot{\gamma}}\delta^\mu{}_\rho \otimes (-\sigma^\sigma_{\delta\dot{\delta}})\bar{\theta}^{\dot{\delta}}\delta^\nu{}_\sigma\right) \\
&= x^\mu \cdot x^\nu + C^{\alpha\beta}\sigma^\mu_{\alpha\dot{\gamma}}\sigma^\nu_{\beta\dot{\delta}}\bar{\theta}^{\dot{\gamma}}\bar{\theta}^{\dot{\delta}},
\end{aligned} \tag{5.24}$$

$$
\begin{aligned}
x^\mu \star \theta^\alpha &= m \circ e^{\frac{1}{2}C^{\gamma\delta}Q_\gamma \otimes Q_\delta}(x^\mu \otimes \theta^\alpha) \\
&= m\left(x^\mu \otimes \theta^\alpha + \frac{1}{2}C^{\gamma\delta}(-\sigma^\rho_{\gamma\dot\gamma}\bar\theta^{\dot\gamma}\delta^\mu{}_\rho) \otimes i\delta^\alpha{}_\delta\right) \\
&= x^\mu \cdot \theta^\alpha - i\frac{1}{2}C^{\alpha\beta}\sigma^\mu_{\beta\dot\gamma}\bar\theta^{\dot\gamma}. \qquad (5.25)
\end{aligned}
$$

This leads nonanticommutative representation of superspace.

$$
\{\theta^\alpha, \theta^\beta\}_\star = C^{\alpha\beta} \qquad (5.26)
$$

We assume here that the indices of the constant is symmetric, $C^{\alpha\beta} = C^{\beta\alpha}$.

In addition to this, the other noncommutative relations are

$$
\begin{aligned}
[x^\mu, x^\nu]_\star &= C^{\alpha\beta}\sigma^\mu_{\alpha\dot\gamma}\sigma^\nu_{\beta\dot\delta}\bar\theta^{\dot\gamma}\bar\theta^{\dot\delta}, \\
[x^\mu, \theta^\alpha]_\star &= -iC^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}\bar\theta^{\dot\beta}.
\end{aligned}
$$

This results are in agreement with the $\mathcal{N} = 1/2$ SUSY case given by Seiberg[19].

We have to note that the product in the last line of Eq.(5.23)-(5.25) is not the object in ordinary superspace, as we mentioned in the previous section. To get ordinary (anti)commutative superspace representation, the following equivalence relations should be imposed.

$$
\begin{cases}
x^\mu \cdot x^\nu - x^\nu \cdot x^\mu = 0 \\
\theta^\alpha \cdot \theta^\beta + \theta^\beta \cdot \theta^\alpha = 0 \\
\bar\theta^{\dot\alpha} \cdot \bar\theta^{\dot\beta} + \bar\theta^{\dot\beta} \cdot \bar\theta^{\dot\alpha} = 0 \\
x^\mu \cdot \theta^\alpha - \theta^\alpha \cdot x^\mu = 0 \\
x^\mu \cdot \bar\theta^{\dot\alpha} - \bar\theta^{\dot\alpha} \cdot x^\mu = 0 \\
\theta^\alpha \cdot \bar\theta^{\dot\beta} + \bar\theta^{\dot\beta} \cdot \theta^\alpha = 0
\end{cases} \qquad (5.27)
$$

Again the dotted product is associated with the product in superspace in proper ordering.

5.2.3 P-Q Twist

We can make the mixed noncommutativity between bosonic and fermionic coordinate. Consider the following twist element,

$$
\mathcal{F}^{PQ} = \exp\left[\frac{i}{2}\lambda^{\mu\alpha}(P_\mu \otimes Q_\alpha - Q_\alpha \otimes P_\mu)\right]. \qquad (5.28)
$$

Where $\lambda^{\mu\alpha}$ should be a *Grassmann number*, to keep the parenthetic argument bosonic.

From this we have the noncommutative relations,

$$\begin{aligned}
&[x^\mu, x^\nu]_\star = \lambda^{\mu\alpha}\sigma^\nu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} - \lambda^{\nu\alpha}\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}, \\
&[x^\mu, \theta^\alpha]_\star = i\lambda^{\mu\alpha}, \\
&\left\{\theta^\alpha, \theta^\beta\right\}_\star = 0.
\end{aligned} \tag{5.29}$$

In fact, either $P_\mu \otimes Q_\alpha$ or $P_\mu \otimes Q_\alpha$ satisfies the conditions for the twist element. We choose the combination of them to make proper commutation relation.

5.2.4 Mixed Twist

Next we will consider more general twist element.

$$\mathcal{F}^{\text{mix}} = \exp\left[\frac{i}{2}\Theta^{\mu\nu}P_\mu \otimes P_\nu + \frac{i}{2}\lambda^{\mu\alpha}(P_\mu \otimes Q_\alpha - Q_\alpha \otimes P_\mu) - \frac{1}{2}C^{\alpha\beta}Q_\alpha \otimes Q_\beta\right] \tag{5.30}$$

This twist element is a compilation of the previous three twist elements. Each element in the argument, $\Theta^{\mu\nu}P_\mu \otimes P_\nu$, $\lambda^{\mu\alpha}(P_\mu \otimes Q_\alpha - Q_\alpha \otimes P_\mu)$ and $C^{\alpha\beta}Q_\alpha \otimes Q_\beta$, all commute with each other, thus $\mathcal{F}^{\text{mix}}$ is factorizable,

$$\mathcal{F}^{\text{mix}} = \mathcal{F}^{PP}\mathcal{F}^{QQ}\mathcal{F}^{PQ}. \tag{5.31}$$

And it gives the following commutator relations.

$$\begin{aligned}
&[x^\mu, x^\nu]_\star = i\Theta^{\mu\nu} + C^{\alpha\beta}\sigma^\mu_{\alpha\dot{\gamma}}\sigma^\nu_{\beta\dot{\delta}}\bar{\theta}^{\dot{\gamma}}\bar{\theta}^{\dot{\delta}} + \lambda^{\mu\alpha}\sigma^\nu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} - \lambda^{\nu\alpha}\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}, \\
&[x^\mu, \theta^\alpha]_\star = i\lambda^{\mu\alpha} - iC^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}, \\
&\left\{\theta^\alpha, \theta^\beta\right\}_\star = C^{\alpha\beta}.
\end{aligned} \tag{5.32}$$

Actually these results are the linear combinations of the commutation relation of the previous three twists.

5.2.5 Algebra Consistency

We verify the consistency of the algebraic structure in our twisted super Poincaré algebra. For that purpose we will calculate the transformation properties of the noncommutativity parameters, as in the section 4.6.

• P-P Twist

For P-P twisted Poincaré algebra, we already showed the twisted Lorentz transformation of the parameter $\Theta^{\mu\nu}$ in the section 4.6. That is all the same in twisted super Poincaré algebra. Since the generator P_μ commute with both Q_α and $\bar{Q}_{\dot\alpha}$, the coproducts of both generators are not changed thus there is no differences.

$$\begin{aligned} M_{\mu\nu}\Theta^{\rho\sigma} &= M_{\mu\nu}(x^\rho\star x^\sigma - x^\sigma\star x^\rho) = 0 \\ Q_\alpha\Theta^{\rho\sigma} &= Q_\alpha(x^\rho\star x^\sigma - x^\sigma\star x^\rho) = 0 \\ \bar{Q}_{\dot\alpha}\Theta^{\rho\sigma} &= \bar{Q}_{\dot\alpha}(x^\rho\star x^\sigma - x^\sigma\star x^\rho) = 0 \end{aligned} \tag{5.33}$$

• Q-Q Twist

In Q-Q twisted super Poincaré algebra, the coproduct of $M_{\mu\nu}$ is changed,

$$\begin{aligned} \Delta_t^{QQ}(M_{\mu\nu}) &= M_{\mu\nu}\otimes\hat{\mathbf{1}} + \hat{\mathbf{1}}\otimes M_{\mu\nu} \\ &\quad + \frac{i}{2}\left(\sigma_{\mu\nu}\right)_\alpha^{\ \gamma} C^{\alpha\beta}\left(Q_\beta\otimes Q_\gamma + Q_\gamma\otimes Q_\beta\right), \end{aligned} \tag{5.34}$$

and the coproduct of $\bar{Q}_{\dot\alpha}$ is changed at the same time,

$$\begin{aligned} \Delta_t^{QQ}(\bar{Q}^{\dot\alpha}) &= \bar{Q}^{\dot\alpha}\otimes\hat{\mathbf{1}} + \hat{\mathbf{1}}\otimes\bar{Q}^{\dot\alpha} \\ &\quad + C^{\gamma\delta}\varepsilon^{\dot\alpha\dot\beta}\{(\sigma^\rho)_{\gamma\dot\beta}P_\rho\otimes Q_\delta - Q_\gamma\otimes(\sigma^\rho)_{\delta\dot\beta}P_\rho\}. \end{aligned} \tag{5.35}$$

After some exercise, we see

$$\begin{aligned} M_{\mu\nu}C^{\alpha\beta} &= M_{\mu\nu}(\theta^\alpha\star\theta^\beta + \theta^\beta\star\theta^\alpha) \\ &= m\circ(F^{QQ})^{-1}\left(\Delta_t^{QQ}(M_{\mu\nu})(\theta^\alpha\otimes\theta^\beta + \theta^\beta\otimes\theta^\alpha)\right) \\ &= 0. \end{aligned} \tag{5.36}$$

In a similar way,

$$Q_\alpha C^{\alpha\beta} = Q_\alpha(\theta^\alpha \star \theta^\beta + \theta^\beta \star \theta^\alpha) = 0 \tag{5.37}$$

$$\bar{Q}_{\dot\alpha} C^{\alpha\beta} = \bar{Q}_{\dot\alpha}(\theta^\alpha \star \theta^\beta + \theta^\beta \star \theta^\alpha) = 0 \tag{5.38}$$

Therefore noncommutativity parameter $\Theta^{\alpha\beta}$ is not transformed under the twisted supersymmetric transformation as well as the twisted Lorentz transformation.

- P-Q Twist

Like as the Q-Q twist case, in P-Q twisted super Poincaré algebra the coproduct of $M_{\mu\nu}$ and $\bar{Q}_{\dot\alpha}$ are changed.

$$\begin{aligned}\Delta_t^{PQ}(M_{\mu\nu}) &= M_{\mu\nu} \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes M_{\mu\nu} \\ &\quad -\frac{1}{2}\lambda^{\rho\alpha}\left[(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \otimes Q_\alpha - Q_\alpha \otimes (\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)\right. \\ &\quad \left. - P_\rho \otimes (\sigma_{\mu\nu})_\alpha{}^\gamma Q_\gamma + (\sigma_{\mu\nu})_\alpha{}^\gamma Q_\gamma \otimes P_\rho\right],\end{aligned} \tag{5.39}$$

$$\begin{aligned}\Delta_t^{PQ}(\bar{Q}^{\dot\alpha}) &= \bar{Q}^{\dot\alpha} \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes \bar{Q}^{\dot\alpha} \\ &\quad + \lambda^{\kappa\gamma}\varepsilon^{\dot\alpha\dot\beta}(\sigma^\rho)_{\gamma\dot\beta}\,(P_\kappa \otimes P_\rho - P_\rho \otimes P_\kappa)\,.\end{aligned} \tag{5.40}$$

Using these coproducts to calculate explicitly, we find

$$\begin{aligned}M_{\mu\nu}(\lambda^{\rho\alpha}) &= m \circ (\mathcal{F}^{PQ})^{-1}\left(\frac{1}{i}\Delta_t(M_{\mu\nu})(x^\rho \otimes \theta^\alpha - \theta^\alpha \otimes x^\rho)\right) \\ &= 0,\end{aligned} \tag{5.41}$$

and

$$Q^\alpha(\lambda^{\rho\alpha}) = Q^\alpha(x^\rho \star \theta^\alpha - \theta^\alpha \star x^\rho) = 0, \tag{5.42}$$

$$\bar{Q}^{\dot\alpha}(\lambda^{\rho\alpha}) = \bar{Q}^{\dot\alpha}(x^\rho \star \theta^\alpha - \theta^\alpha \star x^\rho) = 0. \tag{5.43}$$

- Mixed Twist

In fact, the deformed coproducts in mixed twisted super Poincaré algebra are the linear combination of the previous three twist.

$$\begin{aligned}
\Delta_t^{\rm Mix}(M_{\mu\nu}) &= M_{\mu\nu}\otimes\hat{\mathbf{1}}+\hat{\mathbf{1}}\otimes M_{\mu\nu}\\
&-\frac{1}{2}\Theta^{\rho\sigma}\left[(\eta_{\rho\mu}P_\nu-\eta_{\rho\nu}P_\mu)\otimes P_\sigma+P_\rho\otimes(\eta_{\sigma\mu}P_\nu-\eta_{\sigma\nu}P_\mu)\right]\\
&+\frac{i}{2}\left(\sigma_{\mu\nu}\right)_\alpha^{\ \gamma}C^{\alpha\beta}\left(Q_\beta\otimes Q_\gamma+Q_\gamma\otimes Q_\beta\right)\\
&-\frac{1}{2}\lambda^{\rho\alpha}\left[(\eta_{\mu\rho}P_\nu-\eta_{\nu\rho}P_\mu)\otimes Q_\alpha-Q_\alpha\otimes(\eta_{\mu\rho}P_\nu-\eta_{\nu\rho}P_\mu)\right]\\
&\quad-\left[P_\rho\otimes\left(\sigma_{\mu\nu}\right)_\alpha^{\ \gamma}Q_\gamma+\left(\sigma_{\mu\nu}\right)_\alpha^{\ \gamma}Q_\gamma\otimes P_\rho\right],
\end{aligned}\tag{5.44}$$

$$\begin{aligned}
\Delta_t^{\rm mix}(\bar{Q}^{\dot\alpha}) &= \bar{Q}^{\dot\alpha}\otimes\hat{\mathbf{1}}+\hat{\mathbf{1}}\otimes\bar{Q}^{\dot\alpha}\\
&+\ C^{\gamma\delta}\varepsilon^{\dot\alpha\dot\beta}\{(\sigma^\rho)_{\gamma\dot\beta}P_\rho\otimes Q_\delta-Q_\gamma\otimes(\sigma^\rho)_{\delta\dot\beta}P_\rho\}\\
&+\ \lambda^{\kappa\gamma}\varepsilon^{\dot\alpha\dot\beta}(\sigma^\rho)_{\gamma\dot\beta}\left(P_\kappa\otimes P_\rho-P_\rho\otimes P_\kappa\right).
\end{aligned}\tag{5.45}$$

The transformations of all (anti)commutation relations by nontrivial (deformed coproduct) generators are as follows.

$$\begin{aligned}
m_t^{\rm mix}\left(\Delta_t^{\rm mix}(M_{\mu\nu})(x^\rho\otimes x^\sigma-x^\sigma\otimes x^\rho)\right) &= i\left[\lambda^{\sigma\alpha}(\sigma^\rho)_{\alpha\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\ \dot\gamma}-\lambda^{\rho\alpha}(\sigma^\sigma)_{\alpha\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\ \dot\gamma}\right]\bar\theta^{\dot\gamma},\\
m_t^{\rm mix}\left(\Delta_t^{\rm mix}(M_{\mu\nu})(x^\rho\otimes\theta^\alpha-\theta^\alpha\otimes x^\rho)\right) &= C^{\gamma\alpha}(\sigma^\rho)_{\gamma\dot\kappa}(\bar\sigma_{\mu\nu})^{\dot\kappa}_{\ \dot\gamma}\bar\theta^{\dot\gamma},\\
m_t^{\rm mix}\left(\Delta_t^{\rm mix}(M_{\mu\nu})(\theta^\alpha\otimes\theta^\beta+\theta^\beta\otimes\theta^\alpha)\right) &= 0,
\end{aligned}\tag{5.46}$$

$$\begin{aligned}
m_t^{\rm mix}\left(\Delta_t^{\rm mix}(\bar Q^{\dot\alpha})(x^\rho\otimes x^\sigma-x^\sigma\otimes x^\rho)\right) &= -iC^{\gamma\delta}\varepsilon^{\dot\alpha\dot\beta}\left[(\sigma^\rho)_{\gamma\dot\beta}(\sigma^\sigma)_{\delta\dot\delta}\bar\theta^{\dot\delta}-(\sigma^\rho)_{\gamma\dot\gamma}\bar\theta^{\dot\gamma}(\sigma^\sigma)_{\delta\dot\beta}\right]\\
&\quad-i\varepsilon^{\dot\alpha\dot\beta}\left[\lambda^{\rho\gamma}(\sigma^\sigma)_{\gamma\dot\beta}-\lambda^{\sigma\gamma}(\sigma^\rho)_{\gamma\dot\beta}\right],\\
m_t^{\rm mix}\circ\left(\Delta_t^{\rm mix}(\bar Q^{\dot\alpha})(x^\rho\otimes\theta^\alpha-\theta^\alpha\otimes x^\rho)\right) &= -C^{\gamma\alpha}\varepsilon^{\dot\alpha\dot\beta}(\sigma^\rho)_{\gamma\dot\beta},\\
m_t^{\rm mix}\circ\left(\Delta_t^{\rm mix}(\bar Q^{\dot\alpha})(\theta^\alpha\otimes\theta^\beta+\theta^\beta\otimes\theta^\alpha)\right) &= 0.
\end{aligned}\tag{5.47}$$

These transformed commutation relations by Lorentz generator and antisupercharge are consist with the mixed twisted commutation relations (5.32). In which, all noncommutativity parameters are transformed in Lorentz and supersymmetric invariant way, *i.e.*, like as constants, while all the coordinates of superspace are transformed exactly as the coordinates.

5.3 Central Charge Twist

Extended ($\mathcal{N} \geq 2$) supersymmetric Poincaré algebra modifies the commutators in $\mathcal{N} = 1$ supersymmetric Poincaré algebra in the following way.

$$
\begin{aligned}
[P_\mu, Q^I_\alpha] &= 0, \quad [P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0, \\
\left[M_{\mu\nu}, Q^I_\alpha\right] &= i\left(\sigma_{\mu\nu}\right)_\alpha^{\ \beta} Q^I_\beta, \quad [M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = i\left(\bar{\sigma}_{\mu\nu}\right)^{\dot{\alpha}}_{\ \dot{\beta}} \bar{Q}^{I\dot{\beta}} \\
\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ}, \\
\{Q^I_\alpha, Q^J_\beta\} &= \varepsilon_{\alpha\beta} Z^{IJ}, \quad \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \varepsilon_{\dot{\alpha}\dot{\beta}} Z^*_{IJ}.
\end{aligned}
\tag{5.48}
$$

Here index I and J run from 1 to $\mathcal{N}$. The algebra contains $\mathcal{N}$ supercharges Q^I_α and anti-supercharges $\bar{Q}^I_{\dot{\alpha}}$, and central charges Z^{IJ}.

The extension of the twist procedure to an extended SUSY is not straightforward. In the case of more than one supercharge, they do not form the Abelian algebra because of nonzero central charges $Z^{IJ} \neq 0$. There are few attempts for this problem[36, 37].

Instead of using Q^I_α for the twist, we try to do that in some peculiar way. Pay attention to the fact that Z^{IJ} commute all generators.

$$
\mathcal{F} = \exp\left(\frac{i}{2}\Xi_{IJ}\ Z^I \otimes Z^J\right), \tag{5.49}
$$

where Ξ_{IJ} is a constant. This Z-Z twist clearly satisfies the twist equation.

If the algebra has nonzero central charge, we need extra bosonic coordinate in superspace, namely *central charge coordinate* z^I[28, 29, 30]. Then the central charges are represented as the derivative operators $Z^I = \frac{\partial}{\partial z_I}$, which act on the central charge coordinate space in a way such that

$$
Z^I z_J = \delta^I_J. \tag{5.50}
$$

The twist element (5.49) gives

$$\begin{aligned} z_I \star z_J &= m_t(z_I \otimes z_J) \\ &= m \circ e^{-\frac{i}{2}\Xi_{KL}\ Z^K \otimes Z^L}(z_I \otimes z_J) \\ &= m \circ \left[z_I \otimes z_J + \frac{i}{2}\Xi_{KL}\ \delta_I^K \otimes \delta_J^L \right] \\ &= z_I z_J + \frac{i}{2}\Xi_{IJ}, \end{aligned} \tag{5.51}$$

and the noncommutative central charge coordinate,

$$[z_I, z_J]_\star = i\Xi_{IJ}. \tag{5.52}$$

5.4 Twisted Superconformal Algebra

We have so far seen that we can construct the canonical type noncommutative superspace from the twisted Hopf algebra. It is natural to ask what kind of noncommutativity we can get from the twisted Hopf algebraic procedure. In the following section, we show further studies to get more general noncommutative relations.

To develop the extension of the twisted super Poincaré algebra, superconformal algebra is adopted as the symmetry algebra[43]. The commutation relations of superconformal algebra, and the representations of the generators in superspace are in the appendix B. The twisted conformal algebra is considered in the work[38].

As repeatedly mentioned, an appropriate twist element is easy to construct with the generators in Abelian subalgebra. In superconformal algebra, we can use the P-P, Q-Q and P-Q twist elements as well as super Poincaré algebra, since super Poincaré algebra is included in superconformal algebra as the subalgebra. Moreover, superconformal algebra has more variety of twist, *i.e.*, many Abelian subalgebra. For instance, dilatation generator D itself forms an Abelian subalgebra. In this subsection we will see these possibility separately.

5.4.1 D-D Twist

The D-D twist is a trivial twist.

$$\mathcal{F}^{DD} = \exp(c\, D \otimes D), \tag{5.53}$$

where c is a constant. The twist element changes the multiplication of coordinate space,

$$\begin{aligned} x^\mu \star x^\nu &= m \circ \left((\mathcal{F}^{DD})^{-1} x^\mu \otimes x^\nu \right) \\ &= m \circ \left(e^{-c\, D \otimes D} x^\mu \otimes x^\nu \right) \\ &= e^c x^\mu \cdot x^\nu. \end{aligned} \tag{5.54}$$

Similarly for fermionic coordinate,

$$\theta^\alpha \star \theta^\beta = e^{c/2} \theta^\alpha \cdot \theta^\beta, \quad \bar{\theta}^{\dot{\alpha}} \star \bar{\theta}^{\dot{\beta}} = e^{c/2} \bar{\theta}^{\dot{\alpha}} \cdot \bar{\theta}^{\dot{\beta}}, \quad \text{etc.} \tag{5.55}$$

But no commutator is modified.

5.4.2 S-S Twist

The structure of commutators between K_μ, S_α and $\bar{S}_{\dot{\alpha}}$ is similar to that between P_μ, Q_α and $\bar{Q}_{\dot{\alpha}}$. Thus we can make twist elements from these generators, such as P-Q, Q-Q and P-Q twist.

Most promising twist is the S-S twist,

$$\mathcal{F}_{SS} = \exp\left(-\frac{1}{2} C^{\alpha\beta} S_\alpha \otimes S_\beta \right). \tag{5.56}$$

Owing to the nilpotency of the fermionic generator S_α, the expansion series of the exponential function in Eq.(5.56) terminates in finite number.

If we take the constant $C^{\alpha\beta} = C^{\beta\alpha}$, the twist gives the exotic noncommutative relations between superspace coordinates,

$$\begin{aligned}
[x^\mu, x^\nu]_\star &= -C^{\alpha\gamma} x^\rho x^\sigma \theta^\beta (\sigma^\mu)_{\beta\dot\beta} (\bar\sigma_\rho)^{\dot\beta}{}_\alpha \theta^\delta (\sigma^\nu)_{\delta\dot\delta} (\bar\sigma_\sigma)^{\dot\delta}{}_\gamma, \\
[x^\mu, \theta^\alpha]_\star &= 0, \\
\left[x^\mu, \bar\theta^{\dot\alpha}\right]_\star &= C^{\alpha\beta} \left[-i x^\nu x^\rho \theta^\gamma (\sigma^\mu)_{\gamma\dot\gamma} (\bar\sigma_\nu)^{\dot\gamma}{}_\alpha (\sigma_\rho)_\beta{}^{\dot\alpha} + 2 x^\nu \theta^2 (\bar\sigma^\mu)^{\dot\gamma}{}_\alpha (\sigma_\nu)_\beta^{\{\dot\alpha} \bar\theta^{\dot\gamma\}}\right], \\
\{\theta^\alpha, \theta^\beta\}_\star &= 0, \\
\{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} &= -C^{\alpha\beta} \left[x^\mu x^\nu (\sigma_\mu)_\alpha{}^{\{\dot\alpha} (\sigma_\nu)_\beta{}^{\dot\beta\}} - 4 i x^\mu (\sigma_\mu)_\alpha^{\{\dot\alpha} \theta_\beta \bar\theta^{\dot\beta\}}\right], \\
\{\theta^\alpha, \bar\theta^{\dot\alpha}\}_\star &= -2i C^{\alpha\beta} x^\mu \theta^2 (\sigma_\mu)_\beta{}^{\dot\alpha}.
\end{aligned} \tag{5.57}$$

On the other hand, if we take the constant with antisymmetric indices $C^{\alpha\beta} = -C^{\beta\alpha}$, all the commutators vanish.

To be precise, we have to pay attention to the order of the product. The results of commutators should be written with dotted product although, we wrote them merely as the commutative product in usual superspace here.

5.4.3 K-K Twist and K-S Twist

Next, let us take a look at the K-K Twist,

$$\mathcal{F}^{KK} = \exp\left(-\frac{i}{2}\Theta^{\mu\nu} K_\mu \otimes K_\nu\right). \tag{5.58}$$

Here $\Theta^{\mu\nu}$ is a constant.

This twist element works as well as P-P twist, however, it is difficult to obtain significant noncommutativities from this twist. While K_μ acts on superspace as the derivative operator, it gives higher order terms in x, θ and $\bar\theta$ at the same time. In fact, the commutator $[x^\mu, x^\nu]$ results in infinite series. This endless succession of terms is inevitable, since generator K_μ has the dimension of (length), in contrast to P_μ which has (length)$^{-1}$. Furthermore the representation of K_μ in coordinate space is lengthy and complicated, thus we have not had the complete result. It is difficult even to say whether it can be write in compact formula or not.

The K-S twist is also considered,

$$\mathcal{F}^{KS} = \exp\left[\frac{i}{2}\lambda^{\mu\alpha}(K_\mu \otimes S_\alpha - S_\alpha \otimes K_\mu)\right], \tag{5.59}$$

where $\lambda^{\mu\alpha}$ is anticommutative number. This twist element gives a finite expansion series which is slightly longer than S-S twist. Because of the nilpotency of S_α the expansion series terminate at $\mathcal{O}(\lambda^4)$.

5.5 Non-Abelian twist

Up to now, we have made all the twist elements from the generators in Abelian subalgebra. But that is not necessary. The only points which are essential are that the twist equation and the counit condition are satisfied.

As for the Poincaré algebra, Lukierski *et al.* showed firstly the twist element which uses both P_μ and $M_{\mu\nu}$[39]. They chose a commutative pair of generators , for example, M_{12} and P_3, which commute with each other $[M_{12}, P_3] = 0$. These twists obviously work well.

We found one twist element which is constructed from noncommutative generators[43],

$$\mathcal{F}^{\mathrm{MP}} = \exp\left(c\, \varepsilon^{ijk}(M_{ij} \otimes P_k + P_k \otimes M_{ij})\right), \tag{5.60}$$

where c is some constant, and ε^{ijk} is Levi-Civita symbol,

$$\varepsilon^{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases} . \tag{5.61}$$

The generators which appear in $\mathcal{F}^{\mathrm{MP}}$ are in subalgebra of Poincaré algebra in three dimensional space.

$$\begin{aligned} [M_{ij}, M_{lm}] &= i\delta_{jl}M_{im} - i\delta_{il}M_{jm} - i\delta_{jm}M_{il} + i\delta_{im}M_{jl}, \\ [M_{ij}, P_k] &= -i\delta_{ik}P_j + \delta_{jk}P_i. \end{aligned} \tag{5.62}$$

Here δ_{ij} stands for the Kronecker's delta. Since generators in $\mathcal{F}^{\mathrm{MP}}$ contain M_{12}, M_{23}, M_{31}, P_1 and so on, they do not commute indeed.

It is clear that $\mathcal{F}^{\mathrm{MP}}$ satisfy the counit condition. We can see that it also satisfy the twist equation as follows. The LHS of the twist equation (4.29) is calculated such as,

$$
\begin{aligned}
&\mathcal{F}^{\mathrm{MP}}_{12}(\Delta\otimes\mathrm{id})\mathcal{F}^{\mathrm{MP}} \\
&\quad= \exp\left(c\,\varepsilon^{ijk}(M_{ij}\otimes P_k + P_k\otimes M_{ij})\otimes\hat{\mathbf{1}}\right)\times \\
&\exp\left(c\,\varepsilon^{lmn}(M_{lm}\otimes\hat{\mathbf{1}}\otimes P_n + \hat{\mathbf{1}}\otimes M_{lm}\otimes P_n + P_n\otimes\hat{\mathbf{1}}\otimes M_{lm} + \hat{\mathbf{1}}\otimes P_n\otimes M_{lm})\right).
\end{aligned}
\tag{5.63}
$$

Then we check the commutativity of the argument of the exponential function.

$$
\begin{aligned}
&\big[\varepsilon^{ijk}(M_{ij}\otimes P_k\otimes\hat{\mathbf{1}} + P_k\otimes M_{ij}\otimes\hat{\mathbf{1}}), \\
&\qquad \varepsilon^{lmn}(M_{lm}\otimes\hat{\mathbf{1}}\otimes P_n + \hat{\mathbf{1}}\otimes M_{lm}\otimes P_n + P_n\otimes\hat{\mathbf{1}}\otimes M_{lm} + \hat{\mathbf{1}}\otimes P_n\otimes M_{lm})\big] \\
= &\ \varepsilon^{ijk}\varepsilon^{lmn}\left([M_{ij},M_{lm}]\otimes P_k\otimes P_n + M_{ij}\otimes[P_k,M_{lm}]\otimes P_n\right. \\
&\quad +[M_{ij},P_n]\otimes P_k\otimes M_{lm} + M_{ij}\otimes[P_k,P_n]\otimes M_{lm} \\
&\quad +[P_k,M_{lm}]\otimes M_{ij}\otimes P_n + P_k\otimes[M_{ij},M_{lm}]\otimes P_n \\
&\quad \left.+[P_k,P_n]\otimes M_{ij}\otimes M_{lm} + P_k\otimes[M_{ij},P_n]\otimes M_{lm}\right) \\
= &\ \varepsilon^{ijk}\varepsilon^{lmn}\left((i\delta_{jl}M_{im} - i\delta_{il}M_{jm} - i\delta_{jm}M_{il} + i\delta_{im}M_{jl})\otimes P_k\otimes P_n\right. \\
&+M_{ij}\otimes(i\delta_{lk}P_m - i\delta_{mk}P_l)\otimes P_n + (-i\delta_{in}P_j + i\delta_{jn}P_i)\otimes P_k\otimes M_{lm} \\
&+(i\delta_{lk}P_m - i\delta_{mk}P_l)\otimes M_{ij}\otimes P_n + P_k\otimes(i\delta_{jl}M_{im} - i\delta_{il}M_{jm} - i\delta_{jm}M_{il} + i\delta_{im}M_{jl}) \\
&+P_k\otimes(-i\delta_{in}P_j + i\delta_{jn}P_i)\otimes M_{lm} \\
= &\ 8iM^{mk}\otimes P_k\otimes P_n + (2iM^{mn} - 2iM^{nm})\otimes P_m\otimes P_n + (2iM^{ln} - M^{nl})\otimes P_l\otimes P_n \\
&+P_j\otimes P_k\otimes(-2iM^{jk} + 2iM^{kj}) + P_i\otimes P_k\otimes(-2iM^{ik} + 2iM^{ki}) \\
&+P_m\otimes(+2iM^{mn} - 2iM^{nm})\otimes P_n + P_l\otimes(+2iM^{ln} - 2iM^{nl})\otimes P_n \\
&+P_k\otimes 8iM^{nk}\otimes P_n + P_k\otimes P_j\otimes(-2iM^{jk} + 2iM^{kj}) + P_k\otimes P_i\otimes(-2iM^{ik} + 2iM^{ki}) \\
= &\ 0.
\end{aligned}
\tag{5.64}
$$

We used here $\epsilon^{ijk}\epsilon^{lmn}\delta_{il} = 2\delta^{jm}\delta^{kn} - 2\delta^{jn}\delta^{km}$ and $M^{ij} = -M^{ji}$. In spite of the noncommutativity of the generators which construct the twist element $\mathcal{F}^{\rm MP}$, certain combination of them can commute. Since the commutator gives zero, we can turn the two exponential functions into one freely,

$$\begin{aligned}
& \mathcal{F}^{\rm MP}_{12}(\Delta\otimes{\rm id})\mathcal{F}^{\rm MP} \\
= & \exp\big[c\,\varepsilon^{ijk}\big(M_{ij}\otimes P_k\otimes\hat{\mathbf{1}}+P_k\otimes M_{ij}\otimes\hat{\mathbf{1}} \\
& +M_{lm}\otimes\hat{\mathbf{1}}\otimes P_n+\hat{\mathbf{1}}\otimes M_{lm}\otimes P_n+P_n\otimes\hat{\mathbf{1}}\otimes M_{lm}+\hat{\mathbf{1}}\otimes P_n\otimes M_{lm}\big)\big]
\end{aligned} \tag{5.65}$$

In a similar way we get

$$\begin{aligned}
& \mathcal{F}^{\rm MP}_{23}({\rm id}\otimes\Delta)\mathcal{F}^{\rm MP} \\
= & \exp\big(c\,\varepsilon^{ijk}\hat{\mathbf{1}}\otimes(M_{ij}\otimes P_k+P_k\otimes M_{ij})\big)\times \\
& \exp\big(c\,\varepsilon^{lmn}(M_{lm}\otimes P_n\otimes\hat{\mathbf{1}}+M_{lm}\otimes\hat{\mathbf{1}}\otimes P_n+P_n\otimes M_{lm}\otimes\hat{\mathbf{1}}+P_n\otimes\hat{\mathbf{1}}\otimes M_{lm})\big) \\
= & \exp\big[c\,\varepsilon^{ijk}\big(\hat{\mathbf{1}}\otimes M_{ij}\otimes P_k+\hat{\mathbf{1}}\otimes P_k\otimes M_{ij} \\
& +M_{lm}\otimes P_n\otimes\hat{\mathbf{1}}+M_{lm}\otimes\hat{\mathbf{1}}\otimes P_n+P_n\otimes M_{lm}\otimes\hat{\mathbf{1}}+P_n\otimes\hat{\mathbf{1}}\otimes M_{lm}\big)\big]\,.
\end{aligned} \tag{5.66}$$

Therefore we conclude that $\mathcal{F}^{\rm MP}$ satisfies the twist equation too,

$$\mathcal{F}^{\rm MP}_{12}(\Delta\otimes{\rm id})\mathcal{F}^{\rm MP}=\mathcal{F}^{\rm MP}_{23}({\rm id}\otimes\Delta)\mathcal{F}^{\rm MP}. \tag{5.67}$$

$\mathcal{F}^{\rm MP}$ modifies the product of space coordinate,

$$\begin{aligned}
x^l\star x^m &= m\circ e^{c\,\varepsilon^{ijk}(M_{ij}\otimes P_k+P_k\otimes M_{ij})}(x^l\otimes x^m) \\
&= m\left(x^l\otimes x^m-c\,\varepsilon^{ijk}((x_i\delta_j{}^l-x_j\delta_i{}^l)\otimes\delta_k{}^m+\delta_k{}^l\otimes(x_i\delta_j{}^m-x_j\delta_i{}^m))\right) \\
&= m\left(x^l\otimes x^m-c((\varepsilon^{ilm}x_i-\varepsilon^{ljm}x_j)\otimes\hat{\mathbf{1}}+\hat{\mathbf{1}}\otimes((\varepsilon^{iml}x_i-\varepsilon^{mjl}x_j)\right). \\
&= -4c\varepsilon^{lmi}x_i.
\end{aligned} \tag{5.68}$$

For $c=-\frac{i}{8}$, we get the fuzzy-sphere-like noncommutativity,

$$[x^l,x^m]_\star=i\varepsilon^{lmi}x_i. \tag{5.69}$$

6 Conclusion and Discussion

We have studied the theories in non(anti)commutative superspace and the formulation of noncommutative theories in superspace with a twisted Hopf algebra. Our original works are in the section 3 and 5.

Firstly we have reviewed the usual formulation of noncommutative theories by the Weyl mapping and Moyal product. In particular, an interesting noncommutative superspace, namely the noncommutativity between bosonic and fermionic coordinate in superspace, is investigated in detail. We have found that such noncommutative theory has several unique properties. Through the investigation of the Wess-zumino model, we show the nature of a quantum field theory on such noncommutative superspace. Supersymmetry which the original Wess-zumino model has is broken fully, and the theory turns into $\mathcal{N} = 0$ supersymmetry. Sometimes nilpotency of the noncommutativity parameters greatly simplifies the theory. For example, we can solve the equation of motion of an auxiliary field easily by the method of something like a successive approximation, in spite of the complicated equation of motion. We have also pointed out that the quantum corrections for the vacuum energy are canceled out exactly, at least at the first nontrivial order $\mathcal{O}(\lambda^2, g^2)$, although the theory is no longer supersymmetric.

We have constructed the twisted super Poincaré algebra with a method of the Drinfel'd twist operation in Hopf algebra. Non(anti)commutative superspace of the canonical type is constructed as the representation of the twisted super Poincaré algebra. We have shown that the twist procedure can apply widely to non-canonical type noncommutativities. The twisted superconformal algebra is investigated, which gives a various kind of non(anti)commutative four dimensional superspace. Specifically we have calculated explicitly the noncommutative relations between superspace coordinates, in the case of S-S twist element. For more advanced type of a twist element, we have constructed the twist element from the generators which do not commute with each other, namely P_μ and $M_{\mu\nu}$. That element gives the fuzzy-sphere-like noncommutativity.

In general, all the deformations of algebra are achieved in a *twisted Lorentz invariant* and *twisted supersymmetric* way. As a significant advantage of this formulation, we can start with a well-known and established symmetry algebra and its representation. The only critical step is how to choose an appropriate twist element. The other procedures are almost automatic. The deformation is performed at algebraic level, that is fundamentally independent of representations. Whatever the representation is, this procedure works, thus it gives a firm framework for noncommutative field theories in superspace. We can get rid of ambiguities of the representation. The representation in a commutative theory can be also adopted even if the symmetry is broken, with the deformation of multiplication rule on it.

In physics, symmetry is an important issue when we consider some physical system. It gives a fundamental framework of the formulation, and sometimes greatly simplifies a practical calculation in both perturbative and nonperturbative. However it is believed that introducing the noncommutativity into the theory breaks certain symmetries. The twist construction of noncommutative theory shed light on such a case.

A Notations and Conventions

We summarize our notations used in the thesis, and the conventions of a spinor calculus.

Greek characters in the middle of the alphabet, *i.e.* μ, ν, ρ, σ $\cdots$, denote the indices of spacetime coordinates, run from zero to three. Greek characters from the beginning of the alphabet, *i.e.* α, β, γ, δ $\cdots$, denote spinorial indices, whose value is one or two.

The metric $\eta_{\mu\nu}$ in Minkowski spacetime is

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{A.1}$$

A.1 Spinor and Superspace

We make a spinor index up or down with antisymmetric epsilon tensors $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$.

$$\epsilon^{12} = \epsilon_{21} = +1 \quad \epsilon^{21} = \epsilon_{12} = -1 \tag{A.2}$$

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \qquad \psi^\alpha = \epsilon^{\alpha\beta}\psi^\beta \tag{A.3}$$

$$\bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}} \qquad \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}} \tag{A.4}$$

The abbreviated contraction rules of spinor index are as follows.

$$\psi\chi \equiv \psi^\alpha\chi_\alpha \tag{A.5}$$

$$\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \tag{A.6}$$

especially

$$\theta^2 = \theta^\alpha\theta_\alpha \quad \bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \tag{A.7}$$

Sigma matrix $\sigma^\mu{}_{\alpha\dot{\beta}}$ is a 2×2 matrix to connect spinors coordinate to spacetime coordinate.

$$\sigma^\mu{}_{\alpha\dot{\beta}}\bar{\sigma}^{\nu\dot{\beta}\gamma} + \sigma^\nu{}_{\alpha\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\gamma} = 2\eta_{\mu\nu}\delta_\alpha{}^\gamma \tag{A.8}$$

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.9}$$

$$\bar{\sigma}^{\mu\ \dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^{\mu}{}_{\beta\dot{\beta}} \tag{A.10}$$

Lorentz generators of Weyl spinors are defined with the Sigma matrices.

$$\begin{aligned}(\sigma_{\mu\nu})_{\alpha}^{\ \beta} &= -\frac{1}{4}(\sigma_{\mu}\bar{\sigma}_{\nu} - \sigma_{\nu}\bar{\sigma}_{\mu})_{\alpha}^{\ \beta} \\ (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} &= -\frac{1}{4}(\bar{\sigma}_{\mu}\sigma_{\nu} - \bar{\sigma}_{\nu}\sigma_{\mu})^{\dot{\alpha}}_{\ \dot{\beta}}\end{aligned} \tag{A.11}$$

We define an abbreviation of derivative operator on spacetime.

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} \tag{A.12}$$

The conventions of the derivative operators with Grassmann numbers are here.

$$\frac{\partial}{\partial\theta^{\alpha}}\theta^{\beta} = \delta_{\alpha}^{\ \beta} \qquad \frac{\partial}{\partial\theta_{\alpha}}\theta_{\beta} = \delta^{\alpha}_{\ \beta} \tag{A.13}$$

$$\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\ \dot{\beta}} \qquad \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}_{\dot{\beta}} = \delta^{\dot{\alpha}}_{\ \dot{\beta}} \tag{A.14}$$

The rules (A.13)-(A.14) imply the following.

$$\frac{\partial}{\partial\theta_{\alpha}} = -\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^{\beta}} \quad \frac{\partial}{\partial\theta^{\alpha}} = -\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_{\beta}} \tag{A.15}$$

$$\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} = -\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} \quad \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}} \tag{A.16}$$

- Hermitian conjugation.
  - Hermite conjugate of Weyl spinors.

$$(\psi^{\alpha})^{\dagger} = \bar{\psi}^{\dot{\alpha}} \quad (\psi_{\alpha})^{\dagger} = \bar{\psi}_{\dot{\alpha}} \quad (\bar{\psi}^{\dot{\alpha}})^{\dagger} = \psi^{\alpha} \quad (\bar{\psi}_{\dot{\alpha}})^{\dagger} = \psi_{\alpha} \tag{A.17}$$

These rules leads to

$$(\psi\chi)^{\dagger} = (\chi_{\alpha})^{\dagger}(\psi^{\alpha})^{\dagger} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\psi}\bar{\chi}. \tag{A.18}$$

  - Hermite conjugate of epsilon tensors.

$$(\epsilon^{\alpha\beta})^{\dagger} = \epsilon^{\dot{\alpha}\dot{\beta}} \quad (\epsilon_{\alpha\beta})^{\dagger} = \epsilon_{\dot{\alpha}\dot{\beta}} \tag{A.19}$$

– Hermite conjugate of sigma matrices.

$$(\sigma^{\mu\ \alpha\dot{\beta}})^\dagger = \sigma^{\mu\ \beta\dot{\alpha}} \quad (\bar{\sigma}^{\mu}{}_{\dot{\alpha}\beta})^\dagger = \bar{\sigma}^{\mu}{}_{\alpha\dot{\beta}} \tag{A.20}$$

These lead to

$$\begin{aligned} \{(\sigma_{\mu\nu})_\alpha{}^\beta\}^\dagger &= -(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} \\ \{(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\}^\dagger &= -(\sigma_{\mu\nu})_\alpha{}^\beta \end{aligned} \tag{A.21}$$

– Spacetime derivative ∂_μ is an anti-Hermitian operator $(\partial_\mu)^\dagger = -\partial_\mu$, since the combination $i\partial_\mu$ should be Hermite.

Hermitian conjugation of a derivative with fermionic coordinate in superspace.

$$(\frac{\partial}{\partial\theta^\alpha})^\dagger = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \quad (\frac{\partial}{\partial\theta_\alpha})^\dagger = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} \tag{A.22}$$

Note a sign flip when a coefficient is fermionic, e.g.

$$(\theta^\beta\frac{\partial}{\partial\theta^\alpha})^\dagger = -\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}. \tag{A.23}$$

We summarize some useful equations.

$$\begin{aligned} \theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2 & \theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta^2 \\ \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}^2 & \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^2 \end{aligned} \tag{A.24}$$

$$\begin{aligned} \frac{\partial}{\partial\theta^\alpha}\theta_\beta &= -\varepsilon_{\alpha\beta} & \frac{\partial}{\partial\theta_\alpha}\theta^\beta &= -\varepsilon^{\alpha\beta} \\ \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}_{\dot{\beta}} &= -\varepsilon_{\dot{\alpha}\dot{\beta}} & \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}^{\dot{\beta}} &= -\varepsilon^{\dot{\alpha}\dot{\beta}} \end{aligned} \tag{A.25}$$

$$\begin{aligned} \frac{\partial}{\partial\theta^\alpha}\theta^2 &= 2\theta_\alpha & \frac{\partial}{\partial\theta_\alpha}\theta^2 &= -2\theta^\alpha \\ \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}^2 &= -2\bar{\theta}_{\dot{\alpha}} & \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}^2 &= 2\bar{\theta}^{\dot{\alpha}} \end{aligned} \tag{A.26}$$

$$(\sigma_{\mu\nu})_\alpha{}^\alpha = 0 \tag{A.27}$$

$$\varepsilon_{\beta\gamma}(\sigma_{\mu\nu})_\alpha{}^\gamma = (\sigma_{\mu\nu})_{\alpha\beta} = (\sigma_{\mu\nu})_{\beta\alpha} \qquad (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} = -(\bar{\sigma}_{\mu\nu})_{\dot{\beta}\dot{\alpha}} \tag{A.28}$$

$$\begin{cases} \varepsilon^{\mu\nu\rho\sigma}(\sigma_{\rho\sigma}) = 2i(\sigma^{\mu\nu}) \\ \varepsilon^{\mu\nu\rho\sigma}(\bar{\sigma}_{\rho\sigma}) = -2i(\bar{\sigma}^{\mu\nu}) \end{cases} \begin{cases} \varepsilon_{\mu\nu\rho\sigma}(\sigma^{\rho\sigma}) = 2i(\sigma_{\mu\nu}) \\ \varepsilon_{\mu\nu\rho\sigma}(\bar{\sigma}^{\rho\sigma}) = -2i(\bar{\sigma}_{\mu\nu}) \end{cases} \tag{A.29}$$

Where $\varepsilon^{\mu\nu\rho\sigma}$ and $\varepsilon_{\mu\nu\rho\sigma}$ are anti-symmetric constant tensors, $\varepsilon^{0123} = +1$ and $\varepsilon_{0123} = -1$.

$$\sigma^{\ \mu}_{\alpha\dot{\alpha}}\sigma^{\ \nu}_{\beta\dot{\beta}} - \sigma^{\ \nu}_{\alpha\dot{\alpha}}\sigma^{\ \mu}_{\beta\dot{\beta}} = -2\left[(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\,\varepsilon_{\dot{\alpha}\dot{\beta}} + (\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\,\varepsilon_{\alpha\beta}\right] \tag{A.30}$$

$$\sigma^{\ \mu}_{\alpha\dot{\alpha}}\sigma^{\ \nu}_{\beta\dot{\beta}} + \sigma^{\ \nu}_{\alpha\dot{\alpha}}\sigma^{\ \mu}_{\beta\dot{\beta}} = \eta^{\mu\nu}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} + 4(\sigma^{\rho\mu}\varepsilon)_{\alpha\beta}(\varepsilon\bar{\sigma}^{\rho\nu})_{\dot{\alpha}\dot{\beta}} \tag{A.31}$$

$$\mathrm{Tr}\left(\sigma^{\mu\nu}\sigma^{\rho\sigma}\right) = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) + \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma} \tag{A.32}$$

$$\mathrm{Tr}\left(\sigma_{\mu\nu}\sigma_{\rho\sigma}\right) = \frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}\right) - \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma} \tag{A.33}$$

$$\sigma^\mu\bar{\sigma}^\nu\sigma^\rho + \sigma^\rho\bar{\sigma}^\nu\sigma^\mu = 2(\eta^{\mu\nu}\sigma^\rho + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\rho}\sigma^\nu) \tag{A.34}$$

$$\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho + \bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu = 2(\eta^{\mu\nu}\bar{\sigma}^\rho + \eta^{\nu\rho}\bar{\sigma}^\mu - \eta^{\mu\rho}\bar{\sigma}^\nu) \tag{A.35}$$

$$\sigma^\mu\bar{\sigma}^\nu\sigma^\rho - \sigma^\rho\bar{\sigma}^\nu\sigma^\mu = 2i\varepsilon^{\mu\nu\rho\sigma}\sigma_\sigma \tag{A.36}$$

$$\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho - \bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu = -2i\varepsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma \tag{A.37}$$

$$\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\theta^\beta(\sigma^\nu)_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}} = \eta^{\mu\nu}\theta^2\bar{\theta}^2 \tag{A.38}$$

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\beta}\beta} = 2\delta_\alpha{}^\beta \delta_{\dot{\alpha}}{}^{\dot{\beta}} \tag{A.39}$$

$$\mathrm{Tr}\,(\sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\sigma) = 2\eta^{\mu\nu}\eta^{\rho\sigma} + 2\eta^{\nu\rho}\eta^{\mu\sigma} - 2\eta^{\mu\rho}\eta^{\nu\sigma} + 2i\varepsilon^{\mu\nu\rho\sigma} \tag{A.40}$$

A.2 Noncommutative Parameter

We use the following notations for noncommutativity parameter $\lambda^{\mu\alpha}$.

$$\begin{aligned}
\lambda^{\mu\nu} &\equiv \frac{1}{2}\lambda^{\mu\alpha}\lambda^\nu{}_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta}\lambda^{\mu\alpha}\lambda^{\nu\beta}, \\
\bar{\lambda}^{\mu\nu} &\equiv (\lambda^{\mu\nu})^\dagger = -\frac{1}{2}\bar{\lambda}^{\mu\dot{\alpha}}\bar{\lambda}^{\nu\dot{\beta}}\varepsilon_{\dot{\alpha}\dot{\beta}}, \\
\lambda^{\mu\nu\rho\sigma} &\equiv \frac{1}{4}\lambda^{\mu\nu}\lambda^{\rho\sigma} = \frac{1}{16}\lambda^{\mu\alpha}\lambda^\nu{}_\alpha\lambda^{\rho\beta}\lambda^\sigma{}_\beta, \\
\bar{\lambda}^{\mu\nu\rho\sigma} &\equiv (\lambda^{\mu\nu\rho\sigma})^\dagger = \frac{1}{4}\bar{\lambda}^{\mu\nu}\bar{\lambda}^{\rho\sigma}.
\end{aligned} \tag{A.41}$$

Where upper bar denotes "complex conjugate" of Grassmann number, $\overline{(\lambda^{\mu\alpha})} = \bar{\lambda}^{\mu\dot{\alpha}}$. Do not confuse spacetime indices and spinor indices.

B Superconformal Algebra and Its Representation

$\mathcal{N} = 1$ superconformal algebra consists of translation generators P_μ, Lorentz generators $M_{\mu\nu}$, supercharge Q_α and antisupercharge $\bar{Q}_{\dot{\alpha}}$, special conformal generators K_μ, conformal supercharge S_α and anti-supercharge $\bar{S}_{\dot{\alpha}}$, dilatation generator D. The commutation relations are as follows.

$$
\begin{aligned}
&[P_\mu, P_\mu] = 0 \\
&[M_{\mu\nu}, M_{\rho\sigma}] = i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\mu\rho} \\
&[M_{\mu\nu}, P_\rho] = -i\eta_{\mu\rho}P_\nu + i\eta_{\nu\rho}P_\mu \\
&[P_\mu, Q_\alpha] = 0 \quad [P_\mu, \bar{Q}_{\dot\alpha}] = 0 \\
&[M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \quad [M_{\mu\nu}, \bar{Q}^{\dot\alpha}] = i(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta} \\
&\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu \\
&\{Q_\alpha, Q_\beta\} = 0 \quad \{\bar{Q}^{\dot\alpha}, \bar{Q}^{\dot\beta}\} = 0 \\
&[K_\mu, Q_\alpha] = i(\sigma^\mu)_{\alpha\dot\beta}\bar{S}^{\dot\beta} \quad [K_\mu, \bar{Q}^{\dot\alpha}] = i(\bar\sigma^\mu)^{\dot\alpha\beta}S_\beta \\
&[M_{\mu\nu}, S_\rho] = -i\eta_{\mu\rho}S_\nu + i\eta_{\nu\rho}S_\mu \\
&[K_\mu, K_\nu] = 0 \\
&[P_\mu, K_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}) \\
&\{S_\alpha, \bar{S}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta}K_\mu \\
&\{S_\alpha, S_\beta\} = 0 \quad \{\bar{S}^{\dot\alpha}, \bar{S}^{\dot\beta}\} = 0 \\
&[K_\mu, S_\alpha] = 0 \quad [K_\mu, \bar{S}_{\dot\beta}] = 0 \\
&[M_{\mu\nu}, S_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta S_\beta \quad [M_{\mu\nu}, \bar{S}^{\dot\alpha}] = i(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{S}^{\dot\beta} \\
&[P_\mu, S_\alpha] = i(\sigma^\mu)_{\alpha\dot\beta}\bar{Q}^{\dot\beta} \quad [P_\mu, \bar{Q}^{\dot\alpha}] = i(\bar\sigma^\mu)^{\dot\alpha\beta}Q_\beta \\
&[D, D] = 0 \\
&[D, P_\mu] = -iP_\mu \quad [D, K_\mu] = iK_\mu \\
&[D, Q_\alpha] = -i\frac{1}{2}Q_\alpha \quad [D, S_\alpha] = i\frac{1}{2}S_\alpha \\
&[D, M_{\mu\nu}] = 0
\end{aligned} \tag{B.1}
$$

These generators are represented in the form of the differential operators on superspace.

$$
\begin{aligned}
P_\mu &= i\partial_\mu, \\
M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) - i\theta^\alpha\,(\sigma_{\mu\nu})_\alpha^{\ \beta}\,\frac{\partial}{\partial\theta^\beta} - i\bar\theta_{\dot\alpha}\,(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\,\frac{\partial}{\partial\bar\theta_{\dot\beta}}, \\
Q_\alpha &= i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu, \\
\bar Q_{\dot\alpha} &= -i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \\
D &= ix^\mu\partial_\mu + \frac{i}{2}\theta^\alpha\frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}\bar\theta_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}, \\
K_\mu &= 2ix_\mu x^\nu\partial_\nu - ix^\nu x_\nu\partial_\mu - \theta^2\bar\theta^2\partial_\mu \\
&\quad +2ix^\nu\theta^\alpha(\sigma_{\nu\mu})_\alpha^{\ \beta}\frac{\partial}{\partial\theta^\beta} + ix_\mu\theta^\alpha\frac{\partial}{\partial\theta^\alpha} + \theta^2\bar\theta_{\dot\alpha}\bar\sigma_\mu^{\dot\alpha\alpha}\frac{\partial}{\partial\theta^\alpha} \\
&\quad -2ix^\nu(\bar\sigma_{\nu\mu})^{\dot\alpha}_{\ \dot\beta}\bar\theta^{\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + ix_\mu\bar\theta_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}} - \bar\theta^2\theta_\alpha\sigma_\mu^{\alpha\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}} \\
S_\alpha &= -ix^\mu\theta^\beta(\sigma^\nu)_{\beta\dot\beta}(\bar\sigma_\mu)^{\dot\beta}_{\ \alpha}\partial_\nu - \theta^2(\sigma^\nu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\nu \\
&\quad +2i\theta^2\frac{\partial}{\partial\theta^\alpha} + x^\mu(\sigma_\mu)_\alpha^{\ \dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\beta}} + 2i\theta_\alpha\bar\theta^{\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\beta}} \\
\bar S_{\dot\alpha} &= -ix^\mu(\bar\sigma_\mu)_{\dot\alpha}^{\ \beta}(\sigma^\nu)_{\beta\dot\beta}\bar\theta^{\dot\beta}\partial_\nu + \bar\theta^2\theta^\beta(\sigma^\nu)_{\beta\dot\alpha}\partial_\nu \\
&\quad -2i\theta^2\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + x^\mu(\bar\sigma_\mu)_{\dot\alpha}^{\ \beta}\frac{\partial}{\partial\theta^\beta} + 2i\bar\theta_{\dot\alpha}\theta^\beta\frac{\partial}{\partial\theta^\beta}
\end{aligned}
\tag{B.2}
$$

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